# LATTICE STRUCTURES IN THE IMAGE ALGEBRA AND APPLICATIONS TO IMAGE PROCESSING

By

JENNIFER L. DAVIDSON

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## LIST OF SYMBOLS

| Symbol | Explanation |
|---|---|
| $\mathbf{Z}$ | the set of integers |
| $\mathbf{R}$ | the set of real numbers |
| $\mathbf{R}^+$ | the set of non-negative real numbers |
| $\mathbf{F}$ | an arbitrary value set |
| $\phi$ | the identity element of $\mathbf{F}$ under its group operation |
| $\mathbf{F}^n$ | the Cartesian product of $\mathbf{F}$ |
| $2^S$ | the power set of S (set of all subsets of S) |
| $\emptyset$ | the empty set |
| $\in, \notin, \subset$ | is an element of, is not an element of, is a subset of |
| $\cup, \cap$ | set union, set intersection |
| $f : \mathbf{X} \rightarrow \mathbf{Y}$ | $f$ is a function from $\mathbf{X}$ to $\mathbf{Y}$ |
| $f^{-1}$ | the inverse of function $f$ |
| $\mathbf{F}_{-\infty}$ | the set $\mathbf{F} \cup \{-\infty\}$ |
| $\mathbf{F}_{\pm\infty}$ | the set $\mathbf{F} \cup \{-\infty, +\infty\}$ |
| $\mathbf{F}_{+\infty}$ | the set $\mathbf{F} \cup \{+\infty\}$ |
| $\vee, \wedge$ | maximum, minimum |
| $\mathbf{X} \backslash \mathbf{Y}$ | the set difference of $\mathbf{X}$ and $\mathbf{Y}$ |
| $\mathbf{W}, \mathbf{X}, \mathbf{Y}$ | coordinate sets |
| $\mathbf{w}, \mathbf{x}, \mathbf{y}$ | pixel locations |
| $\mathbf{F}^{\mathbf{X}}$ | the set of all functions from $\mathbf{X}$ to $\mathbf{F}$ |
| $\mathbf{a}, \mathbf{b}, \mathbf{c}$ | images |
| $\mathbf{1} \in \mathbf{F}^{\mathbf{X}}$ | a constant image on $\mathbf{X}$ with values at each coordinate 1 |
| $\mathbf{0} \in \mathbf{F}^{\mathbf{X}}$ | a constant image on $\mathbf{X}$ with values at each coordinate 0 |

| Symbol | Explanation |
|---|---|
| $\mathbf{l} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ | a one-point template from $\mathbf{X}$ to $\mathbf{X}$ with $\mathbf{l_y(x)} = \begin{cases} \phi & \text{if } \mathbf{x} = \mathbf{y} \\ -\infty & \text{otherwise} \end{cases}$ |
| $\Phi \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ | the null template with $\Phi_{\mathbf{y}}(\mathbf{x}) = -\infty$ for all $\mathbf{y} \in \mathbf{Y}, \mathbf{x} \in \mathbf{X}$ |
| $\chi_S(\mathbf{a})$ | the characteristic function over set S of image $\mathbf{a}$ |
| $f(\mathbf{a})$ | the function $f$ induced pointwise over image $\mathbf{a}$ |
| $\mathcal{S}(\mathbf{t_y})$ | the support of template $\mathbf{t} \in (\mathbf{R}^{\mathbf{X}})^{\mathbf{Y}}$ |
| $\mathcal{S}_{-\infty}(\mathbf{t_y})$ | the infinite support of template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ |
| $\mathcal{S}_{+\infty}(\mathbf{t_y})$ | the positive infinite support of template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ |
| $\mathbf{t_y}$ | the image function of template $\mathbf{t}$ at location $\mathbf{y}$ |
| $\mathbf{r}, \mathbf{s}, \mathbf{t}$ | templates |
| $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ | the set of all $\mathbf{F}$ valued templates from $\mathbf{Y}$ to $\mathbf{X}$ |
| $\oplus$ | generalized convolution |
| $\textcircled{\vee}, \textcircled{\wedge}$ | multiplicative maximum, multiplicative minimum |
| $\boxed{\vee}, \boxed{\wedge}$ | additive maximum, additive minimum |
| $\lvert S \rvert$ | the cardinality function, counting the number of elements in set S |
| $\sum \mathbf{a}$ | the sum of all pixel values of the image $\mathbf{a}$ |
| $\bigvee \mathbf{a}$ | the maximum pixel value in image $\mathbf{a}$ |
| $\mathbf{a}^*$ | the additive dual image of the image $\mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}$ |
| $\bar{\mathbf{a}}$ | the multiplicative dual image of the image $\mathbf{a} \in (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}}$ |
| $\mathbf{t}^*$ | the additive dual template of the template $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ |
| $\bar{\mathbf{t}}$ | the multiplicative dual template of the template $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}}$ |
| $\mathbf{t} \in \mathcal{M}_{mn}$ | an m $\times$ n matrix |
| $\mathbf{t}^i$ | the i-th column of the matrix $\mathbf{t}$ |
| $\mathbf{t}_i$ | the i-th row of the matrix $\mathbf{t}$ |
| $\mathbf{t}'$ | the transpose of the matrix $\mathbf{t}$, or the transpose of the template $\mathbf{t}$ |
| $\mathbf{F}_{\pm\infty}$ | a blog with group $\mathbf{F}$ |
| $\mathcal{S}_{-\infty}(\mathbf{t}_i)$ | the infinite support of matrix $\mathbf{t} \in \mathcal{M}_{mn}$ at row i |
| $\mathcal{S}_{+\infty}(\mathbf{t}_i)$ | the infinite positive support of matrix $\mathbf{t} \in \mathcal{M}_{mn}$ at row i |
| $\chi_S^{\infty}(\mathbf{a})$ | the extended characteristic function |
| $\Leftrightarrow$ | if and only if |

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By

Jennifer L. Davidson

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Chairman: Dr. Gerhard X. Ritter
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The research for this dissertation is concerned with the investigation of an algebraic structure, known as image algebra, which is used for expressing algorithms in image processing. The major result of this research is the establishment of a rigorous and coherent mathematical foundation of the subalgebra of the image algebra involving non-linear image transformations. In particular, a classification in the image algebra of a set of non-linear image transformations called lattice transforms is presented, using minimax matrix algebra as a tool. Several applications to image processing problems are discussed. Specifically, in addition to describing several non-linear transform decomposition techniques, the subalgebra is used as a model and a tool for the development of methods to compute lattice transforms locally.

The basic operands and operations of the image algebra and minimax algebra are defined, as well as the relationships between the two algebras. Properties of the minimax algebra including the lattice eigenvalue problem are mapped to the image algebra. Mathematical morphology is shown to be embedded in the image algebra as a special subclass of lattice transforms. Networks of processors are modeled as graphs, and images are represented as functions defined on the nodes of the graph. It is shown that every lattice image-to-image transform can be weakly factored into a product of lattice transformations each of which are implementable on the network if and only if the graph corresponding to the network is strongly connected. Necessary and sufficient conditions are given to decompose a rectangular template into two strip templates. A division algorithm is given which is a generalization of a boolean skeletonizing technique. The transportation problem from linear programming is expressed in the image algebra. A method to produce an image complexity measure is discussed. Most results are given both in image algebra and matrix algebra notation.

# INTRODUCTION

## Background of the Image Algebra

The results presented in this dissertation reflect the ongoing investigation of the structure of the Air Force image algebra, an algebraic structure specifically designed for use in image processing. The idea of establishing a unifying theory for concepts and operations encountered in image and signal processing has been pursued for a number of years now. It was the 1950's work of von Neumann that inspired Unger to propose a "cellular array" machine on which to implement, in parallel, many algorithms for image processing and analysis [1,2]. Among the machines embodying the original automaton envisioned by von Neumann are NASA's massively parallel processor or MPP [3], and the CLIP series of computers developed by M.J.B. Duff and his colleagues [4,5]. A more general class of cellular array computers are pyramids [6] and the Connection Machine, by Thinking Machines Corporation [7].

Many of the operations that cellular array machines perform can be expressed by a set of primitives, or simple elementary operations. One opinion of researchers who design parallel image processing architectures is that a wide class of image transformations can be represented by a small set of basic operations that induce these architectures. G. Matheron and J. Serra developed a set of two primitives that formed the basis for the initial development of a theoretical formalism capable of expressing a large number of algorithms for image processing and analysis. Special purpose parallel architectures were then designed to implement these ideas. Several systems in use today are Matheron and Serra's Texture

Analyzer [8], the Cytocomputer at the Environmental Research Institute of Michigan (ERIM) [9,10], and Martin Marietta's GAPP [11].

The basic mathematical formalism associated with the above cellular architectures are the concepts of pixel neighborhood arithmetic and mathematical morphology. Mathematical morphology is a mathematical structure used in image processing to express image processing transformations by the use of *structuring elements*, which are related to the shape of the objects to be analyzed. The origins of mathematical morphology lie in work done by H. Minkowski and H. Hadwiger on geometric measure theory and integral geometry [12,13,14]. It was Matheron and Serra who used a few of Minkowski's operations as a basis for describing morphological image transformations [15,16], and then implemented their ideas by building the Texture Analyzer System. Some recent research papers on morphological image processing are Crimmins and Brown [17], Haralick, Lee and Shapiro [18], Haralick, Sternberg and Zhuang [19], and Maragos and Schafer [20,21,22].

It was Serra and Sternberg who first unified morphological concepts into an algebraic theory specifically focusing on image processing and image analysis. The first to use the term "Image Algebra" was, in fact, Sternberg [23,24]. Recently, a new theory encompassing a large class of linear and nonlinear systems was put forth by P. Maragos [25]. However, despite these profound accomplishments, morphological methods have some well known limitations. They cannot, with the exception of a few simple cases, express some fairly common image processing techniques such as Fourier-like transformations, feature extraction based on convolution, histogram equalization transforms, chain-coding, and image rotation. At Perkin-Elmer, P. Miller demonstrated that a straightforward and uncomplicated target detection algorithm, furnished by the U.S. Government, could not be expressed using a morphologically based image algebra [26].

The morphological image algebra is built on the Minkowski addition and subtraction of sets [14], and it is this set-theoretic formulation of its basic operations which does not enable mathematical morphology to be used as a basis for a general purpose algebraic based language for digital image processing. These operations ignore the linear domain, transformations between different domains (spaces of different dimensionalities) and transformations between different value sets, e.g. sets consisting of real, complex, or vector valued numbers. The image algebra which was developed at the University of Florida includes these concepts and also incorporates and extends the morphological operations.

## Parallel Image Processing

The processing of images on digital computers requires enormous amounts of time and memory. With the advent of Very Large Scale Integrated (VLSI) circuits, the cellular array of von Neumann became a reality. There are many types of parallel architectures in existence [27], and various ways of categorizing them have been attempted [28]. The general scheme of one popular type of parallel processor is to have many processing elements, or small processors with limited memory, interconnected by communication links. Each processing element can communicate directly with a single controller as well as with a very small number of its neighbors, usually 1 to 8. When the controller gives a signal, all processing elements simultaneously perform some arithmetic and/or logic operation using the values of its neighbors to which it is connected. This type of parallel processor is called a *neighborhood array processor*, as communication links connect the center processor to a small subset of its spatially nearest neighbors. Two typical neighborhood configurations for local interconnection links are given below. The box with the **x** represents the center processor, and the four (or eight) boxes immediately adjacent to **x** represent the four (or eight) processors

with whom **x** can send and receive information via the communication links. The set of pixel locations relative to the center pixel location, **x**, form the *local neighborhood* of **x**.



(a) (b)

**Figure 1.** Two Neighborhood Configurations.
(a) The von Neumann Configuration; (b) The Moore Configuration.

Some of the parallel processors that have been built to implement this type of connection scheme are the MPP, the Distributed Array Processor (ICL DAP) [27,29], the Geometric Arithmetic Parallel Processor (GAPP), and the CLIP4. There are other types of parallel architectures, such as pipeline computers [23] and systolic arrays [30], which differ in construction and implementation of the neighborhood functions. However, the key feature in most of these architectures is that they have a large number of processing elements, each of which communicates directly with only a small subset of the others.

If every value of a transformed image at location **x** involves arithmetically or logically manipulating information only from pixel locations in the local neighborhood of **x**, then the transform is called a *local* transform. Assuming that a transform can be described in a local manner, the amount of time to perform a local operation globally on neighborhood array processors is the amount of time it takes one processor to perform it, often a single clock cycle. Certain image transforms which were previously too computationally intensive can now be implemented on parallel and distributed processors.

In general, image transforms are not local, that is, the calculation of a transformed value may depend on input values which are spatially very distant from the processing element. In order to use parallel processors, the transform must first be decomposed into a product of local transforms. The existence of local decompositions is of theoretical and practical interest, and as such provides the main thrust behind the research in this dissertation.

While such parallel architectures are attractive for use in image processing, much research still needs to be done and implementation techniques developed in order to use the architectures most efficiently.

## Summary of Results

The results in this dissertation stem from an investigation into the image algebra operations of $\boxed{\vee}$, $\textcircled{\vee}$, and $\vee$. A brief description of the image algebra and its use as a model for image processing is presented. A full discussion of the entire image algebra is presented by Ritter et al. [31]. The results given here focus mainly on two non-linear image transform operations whose underlying values have the structure of a lattice. In particular, it is shown that a previously determined, well-defined mathematical structure called the *minimax algebra* can be used to place the study of a wide class of non-linear, lattice-based image transforms on a solid mathematical foundation. We also discuss the mapping of these transforms to certain types of parallel architectures.

It has been well established that the image algebra is capable of expressing all linear transformations [32]. The embedding of linear algebra into the image algebra makes this possible. The major contributions of this thesis are the development of two isomorphisms between the minimax algebra and image algebra which refines the lattice subalgebra of the

image algebra, and the development of new and useful mathematical tools which are of practical use in the area of image processing.

The dissertation is divided into two parts. Part I gives an introduction to the two algebras, the image algebra and minimax algebra. Part II is devoted to presenting new matrix theoretical results which have applications to solving image processing problems. Specifically, Chapter 1 is of an introductory nature, presenting a historical background of the image algebra and a brief discussion of where lattice structures appear to be useful in mathematically characterizing problems in image processing and operations research. Chapter 1 also presents a brief introduction to the image algebra as well as to the minimax algebra. We mention that although vector lattices are contained within the image algebra, they have been investigated [33] and will not be discussed here. The isomorphisms which embed the minimax algebra into the image algebra are given in Chapter 2, and mapping of the minimax algebra properties in image algebra notation are presented in Chapter 3. In Chapter 4 we give the relationship of mathematical morphology to image algebra. In Chapter 5 we present new matrix theoretical results, which have applications to template decomposition. An algorithm similar to the division algorithm for integers is given both in minimax algebra and image algebra notation in Chapter 6. In Chapter 7 we present the formulation of an operations research in image algebra notation, and give an image complexity algorithm. We then present the conclusions and give suggestions for future research after Chapter 7.

## PART I
## LATTICE STRUCTURES IN IMAGE ALGEBRA AND OPERATIONS RESEARCH

The algebraic structures of early image processing languages such as mathematical morphology had no obvious connection with a lattice structure. Those algebras were developed to express binary image manipulation. As the extension to gray valued images developed, the notions of performing maximums and minimums over a set of numbers emerged. Formal links to lattice structures were not developed until very recently [34], including this dissertation. We present a little background in this area, showing how the lattice properties were inherent in the structures being developed.

The algebraic operations developed by Serra and Sternberg are equivalent and based on the operations of Minkowski addition and Minkowski subtraction of sets in $\mathbf{R}^n$. Given A $\subset \mathbf{R}^n$ and B $\subset \mathbf{R}^n$, Minkowski addition is defined by

$$A + B = \{ a + b : a \in A, b \in B \}$$

and Minkowski subtraction is defined by

$$A / B = \overline{\overline{A} + \overline{B}} ,$$

where the bar denotes set complementation. Mathematical morphology was initially developed for boolean image processing, that is, for processing images that have only two values, say 0 and 1. It was eventually extended to include gray-level image processing, that is, images that take on more than two values. The value set underlying the gray value mathematical morphology structure was the set $\mathbf{R}_{-\infty} \equiv \mathbf{R} \cup \{-\infty\}$, the real numbers with $-\infty$ adjoined. Sternberg's functional notation is most often used to express the two morphological operations, as it is simply stated and easy to implement in computer code. The gray value operations of *dilation* and *erosion*, corresponding to Minkowski addition and

subtraction, respectively, are

$$D(x,y) = \max_{i,j} [A(x - i, y - i) + B(i,j)]$$

$$E(x,y) = \min_{i,j} [A(x - i, y - i) - B(-i,-j)]$$

respectively, where A and B are real valued functions on $\mathbf{R}^2$.

As will be shown, mathematical morphology, which uses the lattice $\mathbf{R}_{-\infty}$, is actually a very special subalgebra of the full image algebra. It is well known that $\mathbf{R}_{\pm\infty} \equiv \mathbf{R} \cup \{+\infty, -\infty\}$ is a complete lattice [35]. The lattice structure provides the basis for categorizing certain classes of image processing problems, which is the main subject of this dissertation.

Operations research has long been known for its class of problems in optimization. A certain type of non-linear operations research problems has been the focus of Cuninghame-Green during his research [36,37]. The types of optimization problems that were considered by this author used arithmetic operations different from the usual multiplication and summation. Some machine scheduling and shortest path problems, for example, could be best characterized by a non-linear system utilizing additions and maximums. A monograph entitled *Minimax Algebra* [38] describes a matrix calculus which uses a special case of what is called a *generalized matrix product* [39], where matrices and vectors assume values from a lattice. A few more conditions such as a group operation on the lattice, and the self-duality of the resulting structure, allow Cuninghame-Green to develop a solid mathematical foundation in which to pose a wide variety of operations research questions. It is an interesting and natural link between matrices with values in a lattice and templates in the image algebra which provides the foundation of this dissertation.

# CHAPTER 1
# THE TWO ALGEBRAS

## 1.1. Image Algebra: Basic Definitions and Notation

This section provides the basic definitions and notation that will be used for the image algebra throughout the dissertation. We will define only those image algebra concepts necessary to describe ideas in this document. For a full discourse on all image algebra operands and operations, we refer the reader to a recent publication [31].

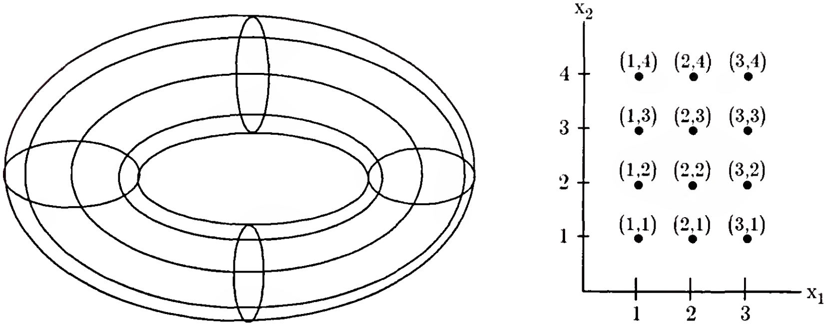
The image algebra is a *heterogeneous algebra*, in the sense of Birkhoff [40], and is capable of describing image manipulations involving not only single valued images, but multivalued images. In fact, it has been formally proven that the set of operations is sufficient for expressing any image-to-image transformation defined in terms of a finite algorithmic procedure, and also that the set of operations is sufficient for expressing any image-to-image transformation for an image which has a finite number of gray values [41,42]. We limit our discussion to single valued images in this document, and refer the reader to other publications on multi-valued images [31].

We will present the six basic operands, some of the finitary operations defined between the operands, and also give a few examples.

### 1.1.1. The Operands of the Image Algebra

The six basic operands are *coordinate sets*, *elements of coordinate sets*, *value sets*, *elements of value sets*, *images*, and *generalized templates*. They are defined as follows.

1. A *coordinate set* $\mathbf{X}$ is a subset of $\mathbf{R}^k$ for some k. Two familiar coordinate sets, the *rectangular* and *toroidal* coordinate sets, are shown in Figure 2.




(a) (b)

**Figure 2.** Two Coordinate Sets.
(a) Toroidal Lattice $\mathbf{X} \subset \mathbf{R}^3$; (b) A Finite Rectangular Array in $\mathbf{R}^2$.

2. A *value set* $\mathbf{F}$ is a semi-group. Some value sets we are interested in are the real numbers, the rational numbers, integers, positive reals, positive rationals, and positive integers. These are denoted by $\mathbf{R}$, $\mathbf{Q}$, $\mathbf{Z}$, $\mathbf{R}^+$, $\mathbf{Q}^+$, and $\mathbf{Z}^+$, respectively. We will also be strongly interested in some of the extended number systems. If $\mathbf{F} \in \{\mathbf{R}, \mathbf{Q}, \mathbf{Z}, \mathbf{R}^+, \mathbf{Q}^+\}$, then $\mathbf{F}_{-\infty}$ denotes $\mathbf{F} \cup \{-\infty\}$, $\mathbf{F}_{+\infty}$ denotes $\mathbf{F} \cup \{+\infty\}$, and $\mathbf{F}_{\pm\infty}$ denotes $\mathbf{F} \cup \{-\infty, +\infty\}$. We denote an arbitrary value set by $\mathbf{F}$.

3. An $\mathbf{F}$ *valued image* $\mathbf{a}$ *on a coordinate set* $\mathbf{X}$ is an element of $\mathbf{F}^{\mathbf{X}}$. Thus, an image $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$ is of form

$$\mathbf{a} = \{(\mathbf{x}, \mathbf{a}(\mathbf{x})) : \mathbf{x} \in \mathbf{X},\ \mathbf{a}(\mathbf{x}) \in \mathbf{F}\ \}.$$

**4.** Let $\mathbf{X}$ and $\mathbf{Y}$ be coordinate sets. An $\mathbf{F}$-*valued template* $\mathbf{t}$ *from* $\mathbf{Y}$ *to* $\mathbf{X}$ is an element of $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$. For each $\mathbf{y} \in \mathbf{Y}$, $\mathbf{t}(\mathbf{y})$ is an image on $\mathbf{X}$. Denoting $\mathbf{t}(\mathbf{y})$ by $\mathbf{t}_{\mathbf{y}}$, we have

$$\mathbf{t}_{\mathbf{y}} = \{ (\mathbf{x}, \mathbf{t}_{\mathbf{y}}(\mathbf{x})) : \mathbf{x} \in \mathbf{X},\ \mathbf{t}_{\mathbf{y}}(\mathbf{x}) \in \mathbf{F} \} \quad \text{for all } \mathbf{y} \in \mathbf{Y}.$$

We give a pictorial representation of a generalized template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ in Figure 3. They are discussed in detailed in the section below on generalized templates.

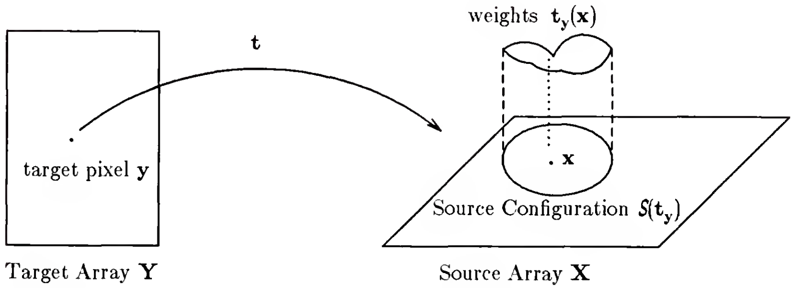



**Figure 3.** A Pictorial Representation of a Generalized Template.

The set $\mathbf{X}$ is called the set of *image coordinates* of $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$, and the range of the function $\mathbf{a}$ is called the *image values* of $\mathbf{a}$. Thus, the image values are a subset of $\mathbf{F}$. The pair $(\mathbf{x},\mathbf{a}(\mathbf{x}))$ is called a *picture element,* or *pixel,* and $\mathbf{x}$ is the *pixel location* of the *pixel value* or *gray value* $\mathbf{a}(\mathbf{x})$. We shall use bold lower case letters, $\mathbf{x}$, to represent a vector in $\mathbf{R}^n$, and lower case letters (not bold) for the components of the vector. Thus $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbf{R}^n$, where $x_i \in \mathbf{R}$ for all i. The set of all $\mathbf{F}$ valued images on $\mathbf{X}$ is denoted by $\mathbf{F}^{\mathbf{X}}$, and the set of all $\mathbf{F}$ valued templates from $\mathbf{Y}$ to $\mathbf{X}$ is denoted by $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$.

As we will not be using any of the operations concerning coordinate sets or value sets, we refer the reader to other publications discussing this topic [31].

1.1.2. Operations on Images

The basic operations on and between $\mathbf{F}$ valued images are the ones induced by the algebraic structure of the value set $\mathbf{F}$. The remaining operations can be defined in terms of these basic ones. In particular, if $\mathbf{F} = \mathbf{R}$, then the basic operations for $\mathbf{a}$, $\mathbf{b} \in \mathbf{R}^{\mathbf{X}}$ are

$$\mathbf{a} + \mathbf{b} \equiv \{\, (\mathbf{x},\mathbf{c}(\mathbf{x})) : \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{b}(\mathbf{x}),\ \mathbf{x} \in \mathbf{X}\}$$

$$\mathbf{a} * \mathbf{b} \equiv \{\, (\mathbf{x},\mathbf{c}(\mathbf{x})) : \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}),\ \mathbf{x} \in \mathbf{X}\}$$

$$\mathbf{a} \vee \mathbf{b} \equiv \{\, (\mathbf{x},\mathbf{c}(\mathbf{x})) : \mathbf{c}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \vee \mathbf{b}(\mathbf{x}),\ \mathbf{x} \in \mathbf{X}\}.$$

If $\mathbf{X}$ is finite, then we define the *dot product* of two images $\mathbf{a}$, $\mathbf{b} \in \mathbf{R}^{\mathbf{X}}$ by

$$\mathbf{a} \bullet \mathbf{b} \;=\; \sum_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) \cdot \mathbf{b}(\mathbf{x}).$$

We say an image $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$ is a *constant* image if its gray value at every pixel location is the same. Thus, a constant image $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$ has form

$$\mathbf{a}(\mathbf{x}) \;=\; k \in \mathbf{F}, \text{ for all } \mathbf{x} \in \mathbf{X}.$$

In this case we write $\mathbf{k}$ for the image $\mathbf{a}$. There are two constant images of importance in the image algebra. One is the *zero* image, defined by $\mathbf{0} \equiv \{(\mathbf{x},0)) : \mathbf{x} \in \mathbf{X}\}$, and the *unit* image, defined by $1 \equiv \{(\mathbf{x},0)) : \mathbf{x} \in \mathbf{X}\}$. These images have the following properties.

$$\mathbf{a} + \mathbf{0} \;=\; \mathbf{0} + \mathbf{a} \;=\; \mathbf{a}$$

$$\mathbf{a} * 1 \;=\; 1 * \mathbf{a} \;=\; \mathbf{a}$$

Suppose f: $\mathbf{F} \rightarrow \mathbf{F}$ is given. Then f induces a function from $\mathbf{F}^{\mathbf{X}}$ to $\mathbf{F}^{\mathbf{X}}$, also called f, where

$$f(\mathbf{a}) \;=\; \{(\mathbf{x},(\mathbf{b}(\mathbf{x})) : \mathbf{b}(\mathbf{x}) = f(\mathbf{a}(\mathbf{x}))\}.$$

For example, the function $f{:}\mathbf{R}\backslash\{0\} \rightarrow \mathbf{R}\backslash\{0\}$ where $f(r) = r^{-1}$ induces a function f: $\mathbf{F}^{\mathbf{X}} \rightarrow \mathbf{F}^{\mathbf{X}}$, where $f(\mathbf{a}) = \mathbf{b}$, and $\mathbf{b}(\mathbf{x}) = 1/\mathbf{a}(\mathbf{x})$, if $\mathbf{a}(\mathbf{x}) \neq 0$, otherwise $\mathbf{b}(\mathbf{x}) = 0$. The image $\mathbf{b}$ so described is denoted by $\mathbf{a}^{-1}$. It is obvious that $\mathbf{a} * \mathbf{a}^{-1} \neq 1$ for every $\mathbf{a}$. But it is true that $\mathbf{a} * \mathbf{a}^{-1} * \mathbf{a} = \mathbf{a}$. For this reason $\mathbf{a}^{-1}$ is called the *pseudo* inverse of $\mathbf{a}$.

If the value set $\mathbf{F} = \mathbf{R}_{\pm\infty}$, then the *additive dual* of $\mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}$ is denoted by $\mathbf{a}^*$ and defined by

$$\mathbf{a}^*(\mathbf{x}) = \begin{cases} -\mathbf{a}(\mathbf{x}) & \text{if } \mathbf{a}(\mathbf{x}) \in \mathbf{R} \\ -\infty & \text{if } \mathbf{a}(\mathbf{x}) = +\infty \\ +\infty & \text{if } \mathbf{a}(\mathbf{x}) = -\infty \end{cases} .$$

Thus we have $(\mathbf{a}^*)^* = \mathbf{a}$.

If $\mathbf{F} = \mathbf{R}_{\pm\infty}^{+}$, then the *multiplicative dual* of $\mathbf{a} \in (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}}$ is denoted by $\bar{\mathbf{a}}$ and defined by

$$\bar{\mathbf{a}}(\mathbf{x}) = \begin{cases} 1/\mathbf{a}(\mathbf{x}) & \text{if } \mathbf{a}(\mathbf{x}) \in \mathbf{R}^+ \\ -\infty & \text{if } \mathbf{a}(\mathbf{x}) = +\infty \\ +\infty & \text{if } \mathbf{a}(\mathbf{x}) = -\infty \end{cases} .$$

It follows that $\overline{(\bar{\mathbf{a}})} = \mathbf{a}$.

Another useful induced function is the *characteristic* function. Let $\chi_T$ denote the usual characteristic function with respect to an arbitrary set T. Here

$$\chi_T(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \notin T \end{cases} .$$

We now define the *generalized* characteristic function of an image $\mathbf{a} \in \mathbf{R}^{\mathbf{X}}$. Let $\mathbf{a} \in \mathbf{R}^{\mathbf{X}}$ and $S \in (2^{\mathbf{F}})^{\mathbf{X}}$. Then the *generalized characteristic function* of an image $\mathbf{a}$ is defined as

$$\chi_S(\mathbf{a}) = \mathbf{c} \in \mathbf{R}^{\mathbf{X}}$$

where

$$\mathbf{c} = \left\{ (\mathbf{x}, \mathbf{c}(\mathbf{x})) : \mathbf{c}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{a}(\mathbf{x}) \in S(\mathbf{x}) \\ 0 & \text{otherwise} \end{cases} \right\}.$$

Note that the usual characteristic function above is a special case of the generalized characteristic function, where $T \subset \mathbf{F}$ and $S(\mathbf{x}) = T$ for all $\mathbf{x} \in \mathbf{X}$. The typical thresholding function applied to an image is a simple example of the generalized characteristic function. Fix $\mathbf{b} \in \mathbf{R}^{\mathbf{X}}$. Then $S_{\leq \mathbf{b}} \in (2^{\mathbf{R}})^{\mathbf{X}}$ is defined by

$$S_{\leq b}(x) \equiv \{r \in \mathbf{R} : r \leq \mathbf{b}(x)\}.$$

To simplify notation, we define $\chi_{\leq \mathbf{b}} \equiv \chi_{S_{\leq \mathbf{b}}}$. Thus, for $\mathbf{b}, \mathbf{a} \in \mathbf{F}^{\mathbf{X}}$, we have

$$\chi_{\leq \mathbf{b}}(\mathbf{a}) = \left\{ (\mathbf{x}, \mathbf{c}(\mathbf{x})) : \mathbf{c}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{a}(\mathbf{x}) \leq \mathbf{b}(\mathbf{x}) \\ 0 & \text{otherwise} \end{cases} \right\}.$$

If we now consider the characteristic function on $\mathbf{R}_{\pm\infty}^{\mathbf{X}}$, we find that we would like our binary output image to have the values $-\infty$ and 0 instead of 0's and 1's, respectively. We define the *extended* characteristic function as the function induced by

$$\chi_S^{\infty}(x) = \begin{cases} 0 & \text{if } x \in S \\ -\infty & \text{otherwise} \end{cases}.$$

Thus, $\chi_{\leq \mathbf{b}}^{\infty}(\mathbf{a})$ is defined as

$$\chi_{\leq \mathbf{b}}^{\infty}(\mathbf{a}) = \begin{cases} 0 & \text{if } \mathbf{a}(\mathbf{x}) \leq \mathbf{b}(\mathbf{x}) \\ -\infty & \text{otherwise} \end{cases}.$$

One unary operation on images is the sum operation, which we will use in Chapter 7. Let $\mathbf{X}$ be a finite coordinate set. Then the *sum* of $\mathbf{a} \in \mathbf{R}^{\mathbf{X}}$ is defined to be

$$\sum \mathbf{a} \equiv \mathbf{a} \bullet \mathbf{1} = \sum_{\mathbf{x} \in \mathbf{X}} \mathbf{a}(\mathbf{x}).$$

In context of the lattice structures of $\mathbf{R}_{\pm\infty}$ and $\mathbf{R}_{\pm\infty}^{+}$, we make the following definition. Let $\mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}$. The *maximum* of $\mathbf{a}$ is the scalar determined by

$$\bigvee \mathbf{a} = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{a}(\mathbf{x}).$$

### 1.1.3. Generalized Templates

For a generalized template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, the coordinate set $\mathbf{Y}$ is called the *target domain* or the domain of $\mathbf{t}$, and $\mathbf{X}$ is called the *range space* of $\mathbf{t}$. The pixel location $\mathbf{y} \in \mathbf{Y}$ at which a template $\mathbf{t}_{\mathbf{y}}$ is evaluated is called a *target point* of the template $\mathbf{t}$, and the values $\mathbf{t}_{\mathbf{y}}(\mathbf{x})$ are called the *weights* of the template $\mathbf{t}$ at $\mathbf{y}$.

If $\mathbf{F} \in \{\mathbf{R}, \mathbf{R}_{\pm\infty}, \mathbf{C}\}$, then for $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, the set

$$\mathcal{S}(\mathbf{t_y}) = \{\mathbf{x} \in \mathbf{X} : \mathbf{t_y}(\mathbf{x}) \neq 0\}$$

is called the *support of* $\mathbf{t_y}$. If $\mathbf{F} \in \{\mathbf{R}_{\pm\infty}, \mathbf{R}^{+}_{\pm\infty}\}$, then for $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ we define

$$\mathcal{S}_{-\infty}(\mathbf{t_y}) = \{\mathbf{x} \in \mathbf{X} : \mathbf{t_y}(\mathbf{x}) \neq -\infty\}$$

and

$$\mathcal{S}_{+\infty}(\mathbf{t_y}) = \{\mathbf{x} \in \mathbf{X} : \mathbf{t_y}(\mathbf{x}) \neq +\infty\}$$

to be the (negative) and positive infinite support, respectively.

If $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}$ and for all triples $\mathbf{x}, \mathbf{y}, \mathbf{z} \in \mathbf{X}$ with $\mathbf{y} + \mathbf{z}$ and $\mathbf{x} + \mathbf{z} \in \mathbf{X}$, we have $\mathbf{t_y}(\mathbf{x}) = \mathbf{t_{y+z}}(\mathbf{x} + \mathbf{z})$, then $\mathbf{t}$ is called *translation invariant*. A template which is not translation invariant is called *translation variant*, or simply variant. Translation invariant templates have the nice property that they may be represented pictorially in a concise manner. The following translation invariant template is presented pictorially in Figure 4. Let $\mathbf{X} = \mathbf{Y} = \mathbf{Z}^2$, and let $\mathbf{y} = (i,j) \in \mathbf{Z}^2$. Let $\mathbf{x}_1 = (i,j)$, $\mathbf{x}_2 = (i+1,j)$, $\mathbf{x}_3 = (i,j-1)$, and $\mathbf{x}_4 = (i+1,j-1)$. Define the weights by $\mathbf{t_y}(\mathbf{x}) = \begin{cases} i & \text{if } \mathbf{x} = \mathbf{x}_i,\ i = 1,\ldots,4 \\ 0 & \text{otherwise} \end{cases}$. Then $\mathcal{S}(\mathbf{t_y}) = \{\mathbf{x}_1, \ldots, \mathbf{x}_4\}$.

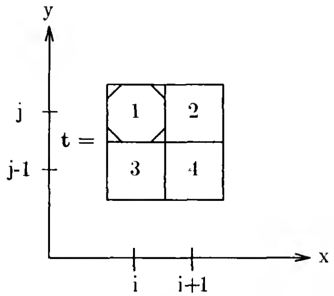



**Figure 4.** Example of a Translation Invariant Template.

The cell with the hash marks in the pictorial representation of $\mathbf{t}$ indicates the location of the target point $\mathbf{y}$.

There are several representations of a template that we will be concerned with. One is the transpose of a template. Let $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$. Then the *transpose* of $\mathbf{t}$ is a template $\mathbf{t}' \in (\mathbf{F}^{\mathbf{Y}})^{\mathbf{X}}$ defined by $\mathbf{t}'_{\mathbf{x}}(\mathbf{y}) \equiv \mathbf{t}_{\mathbf{y}}(\mathbf{x})$. If $\mathbf{F} \in \{\mathbf{R}_{\pm\infty}, \mathbf{R}^{+}_{\pm\infty}\}$, then we can introduce a dual template. For $\mathbf{t} \in (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$, the *additive dual* of $\mathbf{t}$ is the template $\mathbf{t}^* \in (\mathbf{R}^{\mathbf{Y}}_{\pm\infty})^{\mathbf{X}}$ defined by

$$\mathbf{t}^*_{\mathbf{x}}(\mathbf{y}) = \begin{cases} -\mathbf{t}_{\mathbf{y}}(\mathbf{x}) & \text{if } \mathbf{t}_{\mathbf{y}}(\mathbf{x}) \in \mathbf{R} \\ -\infty & \text{if } \mathbf{t}_{\mathbf{y}}(\mathbf{x}) = +\infty. \\ +\infty & \text{if } \mathbf{t}_{\mathbf{y}}(\mathbf{x}) = -\infty \end{cases}$$

Similarly, if $\mathbf{t} \in ((\mathbf{R}^{+}_{\pm\infty})^{\mathbf{X}})^{\mathbf{Y}}$, the *multiplicative dual* of $\mathbf{t}$ is the template $\bar{\mathbf{t}} \in ((\mathbf{R}^{+}_{\pm\infty})^{\mathbf{Y}})^{\mathbf{X}}$ defined by

$$\bar{\mathbf{t}}_{\mathbf{x}}(\mathbf{y}) = \begin{cases} 1/\mathbf{t}_{\mathbf{y}}(\mathbf{x}) & \text{if } \mathbf{t}_{\mathbf{y}}(\mathbf{x}) \in \mathbf{R}^{+} \\ -\infty & \text{if } \mathbf{t}_{\mathbf{y}}(\mathbf{x}) = +\infty. \\ +\infty & \text{if } \mathbf{t}_{\mathbf{y}}(\mathbf{x}) = -\infty \end{cases}$$

### 1.1.4. Operations Between Images and Templates

One common use of templates is to describe some transformation of an input image based on its image values within a subset of the coordinate set $\mathbf{X}$. We first introduce the *generalized product* between an image and a template. Let $\mathbf{X} \subset \mathbf{R}^{n}$ be finite, $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$. Let $\gamma$ be an associative binary operation on the value set $\mathbf{F}$. Then the *global reduce operation* $\Gamma$ *on* $\mathbf{F}^{\mathbf{X}}$ *induced by* $\gamma$ is defined by

$$\Gamma(\mathbf{a}) = \mathop{\Gamma}_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) = \mathbf{a}(\mathbf{x}_1)\gamma\mathbf{a}(\mathbf{x}_2)\gamma \cdots \gamma\mathbf{a}(\mathbf{x}_m),$$

where $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$. Thus, $\Gamma : \mathbf{F}^{\mathbf{X}} \to \mathbf{F}$.

Images and templates are combined by combining appropriate binary operations. Let $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}$ be three value sets, and suppose $o : \mathbf{F}_1 \times \mathbf{F}_2 \rightarrow \mathbf{F}$ and $\hat{o} : \mathbf{F}_2 \times \mathbf{F}_1 \rightarrow \mathbf{F}$ are binary operations. If $\gamma$ is an associative binary operation on $\mathbf{F}$, $\mathbf{a} \in \mathbf{F}_1^{\mathbf{X}}$, and $\mathbf{t} \in (\mathbf{F}_2^{\mathbf{X}})^{\mathbf{Y}}$, then the *generalized backward template operation* of $\mathbf{a}$ *with* $\mathbf{t}$ (*induced by* $\gamma$ *and* $o$) is the binary operation $\textcircled{\gamma} : \mathbf{F}_1^{\mathbf{X}} \times (\mathbf{F}_2^{\mathbf{X}})^{\mathbf{Y}} \rightarrow \mathbf{F}^{\mathbf{Y}}$ defined by

$$\mathbf{a} \textcircled{\gamma} \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \underset{\mathbf{x}\in\mathbf{X}}{\Gamma} \mathbf{a}(\mathbf{x}) \, o \, \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}.$$

If $\mathbf{t} \in (\mathbf{F}_2^{\mathbf{Y}})^{\mathbf{X}}$, then the *generalized forward template operation* of $\mathbf{a}$ *with* $\mathbf{t}$ is defined as

$$\mathbf{t} \textcircled{\gamma} \mathbf{a} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \underset{\mathbf{x}\in\mathbf{X}}{\Gamma} \mathbf{t}_{\mathbf{x}}(\mathbf{y}) \, \hat{o} \, \mathbf{a}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}.$$

Note that the input image $\mathbf{a}$ is an $\mathbf{F}_1$ valued image on the coordinate set $\mathbf{X}$, and the output image $\mathbf{b}$ is an $\mathbf{F}$ valued image on the coordinate set $\mathbf{Y}$, regardless of which template operation, forward or backward, is used. Templates can therefore be used to transform an image on one coordinate set and with values in one set to an image on a completely different coordinate set whose values may be entirely different from the original image's.

Only three special cases of the above generalized operation have been investigated in detail, one by Gader [32] and the other two in this dissertation. Future research will certainly discover other useful combinations. These three operations are denoted by $\oplus$, $\boxed{\vee}$, and $\textcircled{\vee}$. The operation $\oplus$ is a linear one, and we refer the interested reader to other references for recent research in this area [32,43,44]. The other two operations, $\boxed{\vee}$ and $\textcircled{\vee}$, are nonlinear, and investigation of the structure they induce on images and templates is the focus of this dissertation.

Since our main interest concerns the operations $\boxed{\vee}$ and $\textcircled{\vee}$, we will omit the definition $\oplus$ and refer the interested reader to another reference [31]. Let $\mathbf{X} \subset \mathbf{R}^n$ be finite and $\mathbf{Y} \subset \mathbf{R}^m$. Let $\mathbf{a} \in \mathbf{R}_{-\infty}^{\mathbf{X}}$ and $\mathbf{t} \in (\mathbf{R}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}$. Then the *backward additive max* is defined as

$$\mathbf{a} \boxtimes \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\},$$

where $\bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}(\mathbf{x}) = \max\{\, \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}(\mathbf{x}) : \mathbf{x} \in \mathbf{X}\,\}$.

For $\mathbf{t} \in (\mathbf{R}_{-\infty}^{\mathbf{Y}})^{\mathbf{X}}$ we define the *forward additive max* transform by

$$\mathbf{t} \boxtimes \mathbf{a} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{x}}(\mathbf{y}),\ \mathbf{y} \in \mathbf{Y}\}.$$

We use the usual extended arithmetic addition $r + -\infty = -\infty + r = -\infty\ \forall\ r \in \mathbf{R}_{-\infty}$ to define $\mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}(\mathbf{x})$ everywhere.

For $\mathbf{a} \in \mathbf{R}_{-\infty}^{\mathbf{X}}$ and $\mathbf{t} \in ((\mathbf{R}_{-\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}}$ we define the *backward multiplicative max* transform

$$\mathbf{a} \circledvee \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) \cdot \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}.$$

The *forward multiplicative max* transform is given by

$$\mathbf{t} \circledvee \mathbf{a} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) \cdot \mathbf{t}_{\mathbf{x}}(\mathbf{y}),\ \mathbf{y} \in \mathbf{Y}\},$$

where $\mathbf{t} \in ((\mathbf{R}_{-\infty}^{+})^{\mathbf{Y}})^{\mathbf{X}}$.

Recall that a *lattice-ordered group*, or l-group, is a group which is also a lattice. The operation addition (multiplication) on the l-group $\mathbf{R}$ ($\mathbf{R}^{+}$) can be extended in a well-defined manner to addition (multiplication) on $\mathbf{R}_{\pm\infty}$ ($\mathbf{R}_{\pm\infty}^{+}$) by defining

$$x \times -\infty = -\infty \times x = -\infty,\ x \in G \cup \{-\infty\}$$

$$x \times +\infty = +\infty \times x = +\infty,\ x \in G \cup \{+\infty\}$$

$$-\infty \times +\infty = +\infty \times -\infty = -\infty$$

where $\times \in \{+, *\}$, depending on whether $G = \mathbf{R}$ or $\mathbf{R}^{+}$, respectively. Of course, the elements $+\infty$, $-\infty$ have no additive inverse under the operation $+$ or $*$, and hence $\mathbf{R}_{\pm\infty}$ (or $\mathbf{R}_{\pm\infty}^{+}$) is no longer a group. This is discussed in detail in section 1.2, where the notion of a *bounded lattice ordered group*, an extension of a lattice-ordered group with extended

arithmetic, is introduced. This provides for the value set $\mathbf{R}_{\pm\infty}$ to be used in the definition of the image-template operation $\boxtimes$, for example, and the value set $\mathbf{R}_{\pm\infty}^{+}$ to be used in the definition of the image-template operation $\circledvee$.

We remark that for computational as well as theoretical purposes, we can restate the above two convolutions with the new pixel value calculated only over the support of the template $\mathbf{t}$. If $\mathcal{S}_{-\infty}(\mathbf{t_y}) \neq \emptyset$, then $\bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x})+\mathbf{t_y}(\mathbf{x}) = \bigvee_{\mathbf{x}\in\mathcal{S}_{-\infty}(\mathbf{t_y})} \mathbf{a}(\mathbf{x})+\mathbf{t_y}(\mathbf{x})$, and we have

$$\mathbf{a} \boxtimes \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x}\in\mathcal{S}_{-\infty}(\mathbf{t_y})} \mathbf{a}(\mathbf{x})+\mathbf{t_y}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}.$$

Similarly, if $\mathcal{S}_{-\infty}(\mathbf{t_y}) \neq \emptyset$, then $\bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x})*\mathbf{t_y}(\mathbf{x}) = \bigvee_{\mathbf{x}\in\mathcal{S}_{-\infty}(\mathbf{t_y})} \mathbf{a}(\mathbf{x})*\mathbf{t_y}(\mathbf{x})$, and

$$\mathbf{a} \circledvee \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x}\in\mathcal{S}_{-\infty}(\mathbf{t_y})} \mathbf{a}(\mathbf{x})*\mathbf{t_y}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}.$$

If in either case $\mathcal{S}_{-\infty}(\mathbf{t_y}) = \emptyset$, then we define

$$\bigvee_{\mathbf{x}\in\mathcal{S}_{-\infty}(\mathbf{t_y})} \mathbf{a}(\mathbf{x})+\mathbf{t_y}(\mathbf{x}) \quad \text{or} \quad \bigvee_{\mathbf{x}\in\mathcal{S}_{-\infty}(\mathbf{t_y})} \mathbf{a}(\mathbf{x})*\mathbf{t_y}(\mathbf{x}) \quad = -\infty \quad .$$

We may therefore restrict our computation of the new pixel value to the infinite support of $\mathbf{t_y}$. This becomes particularly important when considering mapping of transforms to certain types of parallel architectures, as will be discussed in the introductory remarks to Part II, and Chapter 5.

Because of the duality inherent in the two structures $\mathbf{R}_{\pm\infty}$ and $\mathbf{R}_{\pm\infty}^{+}$, the operations $\boxtimes$ and $\circledvee$ induce dual image-template operations, called *additive minimum* and *multiplicative minimum*, respectively. They are defined by

$$\mathbf{a} \boxdot \mathbf{t} \equiv (\mathbf{t}^* \boxtimes \mathbf{a}^*)^*$$

and

$$\mathbf{a} \circledwedge \mathbf{t} \equiv \overline{(\bar{\mathbf{t}} \circledvee \bar{\mathbf{a}})}.$$

Equivalently, we have

$$\mathbf{a} \boxed{\wedge} \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigwedge_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) +' \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}$$

$$= \{(\mathbf{y},\mathbf{b}(\mathbf{y}) : \mathbf{b}(\mathbf{y}) = \bigwedge_{\mathbf{x}\in S_{+\infty}(\mathbf{t}_{\mathbf{y}})} \mathbf{a}(\mathbf{x}) +' \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}$$

and

$$\mathbf{t} \circledwedge \mathbf{a} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigwedge_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) *' \mathbf{t}_{\mathbf{x}}(\mathbf{y}),\ \mathbf{y} \in \mathbf{Y}\}$$

$$= \bigwedge_{\mathbf{x}\in S_{+\infty}(\mathbf{t}_{\mathbf{x}})} \mathbf{a}(\mathbf{x}) *' \mathbf{t}_{\mathbf{x}}(\mathbf{y}),\ \mathbf{y} \in \mathbf{Y}\}.$$

where the dual operations $+'$ and $*'$ are presented in section 1.2. As before, if $S_{+\infty}(\mathbf{t}_{\mathbf{y}}) = \emptyset$, we define

$$\bigwedge_{\mathbf{x}\in S_{+\infty}(\mathbf{t}_{\mathbf{x}})} \mathbf{a}(\mathbf{x}) +' \mathbf{t}_{\mathbf{x}}(\mathbf{y}) \quad \text{or} \quad \bigwedge_{\mathbf{x}\in S_{+\infty}(\mathbf{t}_{\mathbf{x}})} \mathbf{a}(\mathbf{x}) *' \mathbf{t}_{\mathbf{x}}(\mathbf{y}) \quad = +\infty\,.$$

The above definitions assume that the support $S_{-\infty}(\mathbf{t}_{\mathbf{y}})$ is finite for each $\mathbf{y} \in \mathbf{Y}$. We may extend the above definitions to continuous functions $\mathbf{a}$ and $\mathbf{t}_{\mathbf{y}}$ on a compact set $S_{-\infty}(\mathbf{t}_{\mathbf{y}})$. This is well-defined as the sum or product of two continuous functions on a compact subset of $\mathbf{R}^n$, which is continuous, always contains a maximum. Extending the basic properties of the image algebra operations involving $\boxed{\vee}$ and $\circledvee$ from the discrete case to the continuous case should present little difficulty, and remains an open problem at this time.

## 1.1.5. Operations Between Generalized Templates

The pointwise operations of the value set $\mathbf{F}$ can also be extended to to operations between templates. For example, if $\mathbf{F} = \mathbf{R}$, then we have

$$\mathbf{s} + \mathbf{t} \ \equiv\ \mathbf{r}\text{, where } \mathbf{r}_{\mathbf{y}} = \mathbf{s}_{\mathbf{y}} + \mathbf{t}_{\mathbf{y}}$$

$$\mathbf{s} * \mathbf{t} \ \equiv\ \mathbf{r}\text{, where } \mathbf{r}_{\mathbf{y}} = \mathbf{s}_{\mathbf{y}} * \mathbf{t}_{\mathbf{y}}$$

$$\mathbf{s} \vee \mathbf{t} \ \equiv\ \mathbf{r}\text{, where } \mathbf{r}_{\mathbf{y}} = \mathbf{s}_{\mathbf{y}} \vee \mathbf{t}_{\mathbf{y}}\,.$$

If $\mathbf{F} = \mathbf{R}_{\pm\infty}$, then we define

$$\mathbf{s} + \mathbf{t} \ \equiv \ \mathbf{r}\,, \text{ where } \mathbf{r}_\mathbf{y}(\mathbf{x}) = \begin{cases} \mathbf{s}_\mathbf{y}(\mathbf{x}) + \mathbf{t}_\mathbf{y}(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y}) \cap \mathcal{S}_{-\infty}(\mathbf{s}_\mathbf{y}) \\ \mathbf{s}_\mathbf{y}(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{S}_{-\infty}(\mathbf{s}_\mathbf{y}) \setminus \mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y}) \\ \mathbf{t}_\mathbf{y}(\mathbf{x}) & \text{if } \mathbf{x} \in \mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y}) \setminus \mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y}) \\ -\infty & \text{otherwise} \end{cases}.$$

Note that in the case where $\mathbf{s}$ and $\mathbf{t}$ have no values of $-\infty$ or $+\infty$ anywhere, then the definition of $\mathbf{s} + \mathbf{t}$ on the value set $\mathbf{R}_{\pm\infty}$ degenerates to the definition of $\mathbf{s} + \mathbf{t}$ on the value set $\mathbf{R}$.

The generalized image-template operation $\circledcirc$ generalizes to a generalized template-template product. Let $\mathbf{X} \subset \mathbf{R}^n$ be finite, $\mathbf{X} = \{\mathbf{x}_1, \ldots, \mathbf{x}_m\}$, and let $\gamma$ be an associative binary operation on the value set $\mathbf{F}$ with global reduce operation $\Gamma$ on $\mathbf{F}^\mathbf{X}$. Let $\mathbf{F}_1$, $\mathbf{F}_2$, and $\mathbf{F}$ be three value sets, and suppose $o : \mathbf{F}_1 \times \mathbf{F}_2 \to \mathbf{F}$ is a binary operation. If $\gamma$ is an associative binary operation on $\mathbf{F}$, $\mathbf{t} \in (\mathbf{F}_1^\mathbf{X})^\mathbf{W}$, and $\mathbf{s} \in (\mathbf{F}_2^\mathbf{W})^\mathbf{Y}$, then the *generalized template operation* of $\mathbf{t}$ *with* $\mathbf{s}$ (*induced by* $\gamma$ *and* $o$) is the binary operation $\circledcirc : (\mathbf{F}_1^\mathbf{X})^\mathbf{W} \times (\mathbf{F}_2^\mathbf{W})^\mathbf{Y}$ defined by

$$\mathbf{t} \circledcirc \mathbf{s} \equiv \mathbf{r} \in (\mathbf{F}^\mathbf{X})^\mathbf{Y}, \text{ where}$$

$$\mathbf{r}_\mathbf{y}(\mathbf{x}) \ = \ \Gamma_{\mathbf{w}\in\mathbf{W}} \mathbf{t}_\mathbf{w}(\mathbf{x})\, o\, \mathbf{s}_\mathbf{y}(\mathbf{w})\ ,\ \mathbf{y} \in \mathbf{Y}, \mathbf{x} \in \mathbf{X}.$$

Note that if $|\mathbf{X}| = 1$, then the definition of the generalized template operation of $\mathbf{t}$ and $\mathbf{s}$ degenerates to the definition of the generalized backward template operation of the image $\mathbf{t} \in \mathbf{F}_1^\mathbf{W}$ with the template $\mathbf{s} \in (\mathbf{F}_2^\mathbf{W})^\mathbf{Y}$, and $\mathbf{r} \in \mathbf{F}^\mathbf{Y}$. If $|\mathbf{Y}| = 1$, then the definition of the generalized template operation of $\mathbf{t}$ and $\mathbf{s}$ degenerates to the definition of the forward template operation of the image $\mathbf{s} \in \mathbf{F}_2^\mathbf{W}$ with the template $\mathbf{t} \in (\mathbf{F}_1^\mathbf{X})^\mathbf{W}$, where $\mathbf{r} \in \mathbf{F}^\mathbf{X}$.

The specific cases for $\circledcirc = \oplus$, $\boxtimes$, or $\circledvee$ thus generalize to operations between templates. We give the definitions for $\boxtimes$ and $\circledvee$, and refer the reader to another reference for

the definition of $\oplus$ [31]. Let $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ and $\mathbf{s} \in (\mathbf{R}_{\pm\infty}^{\mathbf{W}})^{\mathbf{X}}$. Then $\mathbf{s} \boxed{\vee} \mathbf{t} = \mathbf{r} \in (\mathbf{R}_{\pm\infty}^{\mathbf{W}})^{\mathbf{Y}}$ is defined by

$$\mathbf{r_y}(\mathbf{w}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{t_y}(\mathbf{x}) + \mathbf{s_x}(\mathbf{w}), \text{ where } \mathbf{w} \in \mathbf{W}.$$

Again, as in the image-template operations, we may restrict our computation to a subset of $\mathbf{X}$. In particular, for $\mathbf{y} \in \mathbf{Y}$, we define the set

$$S_{-\infty}(\mathbf{w}) = \{\mathbf{x} \in \mathbf{X} : \mathbf{x} \in \mathcal{S}_{-\infty}(\mathbf{t_y}) \text{ and } \mathbf{w} \in \mathcal{S}_{-\infty}(\mathbf{s_x})\}.$$

Then $\mathbf{r} = \mathbf{s} \boxed{\vee} \mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{W}})^{\mathbf{Y}}$ is defined by

$$\mathbf{r_y}(\mathbf{w}) = \bigvee_{\mathbf{x} \in S_{-\infty}(\mathbf{w})} \mathbf{t_y}(\mathbf{x}) + \mathbf{s_x}(\mathbf{w}),$$

where we define $\bigvee_{\mathbf{x} \in S_{-\infty}(\mathbf{w})} \mathbf{t_y}(\mathbf{x}) + \mathbf{s_x}(\mathbf{w}) = -\infty$ whenever $S_{-\infty}(\mathbf{w}) = \emptyset$.

The operation $\textcircled{\vee}$ has a similar situation. We have $\mathbf{r} = \mathbf{s} \textcircled{\vee} \mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{W}})^{\mathbf{Y}}$ which is defined by

$$\mathbf{r_y}(\mathbf{w}) = \bigvee_{\mathbf{x} \in S(\mathbf{w})} \mathbf{t_y}(\mathbf{x}) \cdot \mathbf{s_x}(\mathbf{w}),$$

where we define $\bigvee_{\mathbf{x} \in S(\mathbf{w})} \mathbf{t_y}(\mathbf{x}) \cdot \mathbf{s_x}(\mathbf{w}) = -\infty$ whenever $S(\mathbf{w}) = \emptyset$.

It follows from these definitions that the infinite support of the template $\mathbf{r}$ is $\mathcal{S}_{-\infty}(\mathbf{r_y}) = \{\mathbf{w} \in \mathbf{W} : S_{-\infty}(\mathbf{w}) \neq \emptyset\}$.

The definitions given in this section are the elemental ones. Further definitions that play important parts in the theoretical development of the lattice structure of the image algebra will be presented as needed.

We define the complementary operations of $\boxed{\wedge}$ and $\textcircled{\wedge}$ for templates in the natural way. Let $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ and $\mathbf{s} \in (\mathbf{R}_{\pm\infty}^{\mathbf{W}})^{\mathbf{X}}$. Then $\mathbf{s} \boxed{\wedge} \mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{W}})^{\mathbf{Y}}$ is defined by

$$\mathbf{s} \boxed{\wedge} \mathbf{t} \equiv (\mathbf{t}^* \boxed{\vee} \mathbf{s}^*)^*$$

Similarly, for $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}}$ and $\mathbf{s} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{W}})^{\mathbf{X}}$, $\mathbf{s} \circledwedge \mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{W}})^{\mathbf{Y}}$ is defined by

$$\mathbf{s} \circledwedge \mathbf{t} \equiv \overline{(\bar{\mathbf{t}} \circledvee \bar{\mathbf{s}})}.$$

We would like to remark upon one notational deviation between the *Overview*'s [31] definition for the $\circledvee$ operations and the one presented here. Let $\mathbf{R}_{+\infty}^{\geq 0} = \{r \in \mathbf{R} : r \geq 0\} \cup \{+\infty\}$. In the *Overview*, for $\mathbf{a} \in (\mathbf{R}_{+\infty}^{\geq 0})^{\mathbf{X}}$ and $\mathbf{t} \in ((\mathbf{R}_{+\infty}^{\geq 0})^{\mathbf{X}})^{\mathbf{Y}}$, the *backward multiplicative max transform* is defined as

$$\mathbf{a} \circledvee \mathbf{t} \equiv \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{a}(\mathbf{x}) \cdot \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}$$

which is equivalent to

$$\mathbf{a} \circledvee \mathbf{t} = \{(\mathbf{y},\mathbf{b}(\mathbf{y})) : \mathbf{b}(\mathbf{y}) = \bigvee_{\mathbf{x} \in \mathcal{S}(\mathbf{t}_{\mathbf{y}})} \mathbf{a}(\mathbf{x}) \cdot \mathbf{t}_{\mathbf{y}}(\mathbf{x}),\ \mathbf{y} \in \mathbf{Y}\}$$

with $\mathbf{b}(\mathbf{y}) = 0$ if $\mathcal{S}(\mathbf{t}_{\mathbf{y}}) = \emptyset$. The difference between the definition given earlier and this one is the *value set*, namely $\mathbf{R}_{\pm\infty}^{+}$ in this document and $\mathbf{R}_{+\infty}^{\geq 0}$ in the *Overview*. The number 0 acts as a lower bound in $\mathbf{R}_{+\infty}^{\geq 0}$ exactly as $-\infty$ acts as a lower bound in $\mathbf{R}_{\pm\infty}^{+}$. Multiplication of the element 0 with the element $\infty$ follows the same rules as multiplication of the element $-\infty$ with the element $\infty$ as given on page 18. In other words, the element 0 can replace *symbolically* the element $-\infty$. The main advantage of using the number 0 instead of the symbol $-\infty$ is for ease of machine and software implementation. Most real image processing data will have no values corresponding to the symbol $+\infty$, and quite often have non-negative values, including 0's. Using 0 as the bottom element enables that value to be represented easily in the computer, while special programming methods would have to be considered to represent the symbol $-\infty$. For purposes which will become clear in the course of this document, we have remained with the notation $\mathbf{R}_{\pm\infty}^{+}$. In implementing any of the ideas in this dissertation, if the value set at hand is $\mathbf{R}_{\pm\infty}^{+}$, it should be clear that the symbol $-\infty$ can be replaced with a 0 and $\mathcal{S}_{-\infty}(\mathbf{t}(\mathbf{y}))$ replaced by $\mathcal{S}(\mathbf{t}(\mathbf{y}))$, so that representation in computers may be more easily accomplished.

## 1.2 Minimax Algebra

The last 40 years have seen a number of different authors discover, apparently independently, a non-linear algebraic structure, which each has used to solve a different type of problem. The operands of this algebra are the real numbers, with $-\infty$ (or $+\infty$ adjoined), with the two binary operations of addition and maximum (or minimum). The extension of this structure to matrices was formalized mathematically, in the environment in which the above problems were posed, by Cuninghame-Green in his book *Minimax Algebra* [38]. It is well known that the structure of $\mathbf{R}$ with the operations of $+$ and $\vee$ is a semi-lattice ordered group, and that $(\mathbf{R}, \vee, \wedge, +)$ is a lattice-ordered group, or an l-group [35]. Viewing $\mathbf{R}_{-\infty}$ as a set with the two binary operations of $+$ and $\vee$, and then investigating the structure of the set of all n x n matrices with values in $\mathbf{R}_{-\infty}$, leads to an entirely different perspective of a class of non-linear operators. These ideas were applied by Shimbel [45] to communications networks. Two authors, Cuninghame-Green [36,37] and Giffler [46] applied them to the problem of machine-scheduling. Others [47,48,49,50] have discussed their usefulness in applications to shortest path problems in graphs. Cuninghame-Green gives several examples throughout his book [38], primarily in the field of operations research. Another useful application, to image algebra, was again independently developed by G.X Ritter et al. [51].

In fact, the notion of a matrix product can be generalized to what is called the *generalized matrix product* [39], whose definition is given below.

Let $\mathbf{F}$ denote a set of numbers. Let $f$ and $g$ be functions from $\mathbf{F} \times \mathbf{F}$ into $\mathbf{F}$. For simplicity, assume the binary operation $f$ to be associative. Let $\mathbf{F}^{mp}$ denote the set of all m x p matrices with values in $\mathbf{F}$, and let $(a_{ij}) = A \in \mathbf{F}^{mp}$ and $(b_{jk}) = B \in \mathbf{F}^{pn}$. Define $f \cdot g$ to be the function from $\mathbf{F}^{mp} \times \mathbf{F}^{pn}$ into $\mathbf{F}^{mn}$ given by

$$(f \cdot g)(\mathrm{A,B}) \;=\; \mathrm{C},$$

where $c_{ik} = (a_{i1}\, g\, b_{1k}) f (a_{i2}\, g\, b_{2k}) f \cdots f (a_{ip}\, g\, b_{pk})$, for i =1,...,m, k = 1,...,n, and $f$ and $g$ are viewed as binary operations.

Thus, if $f$ denotes addition and $g$ multiplication, then $(f \cdot g)$(A,B) is the ordinary matrix product of matrices A and B. Cuninghame-Green develops the setting for a formal matrix calculus based on the two binary operations $+$ and $\vee$ of the extended real numbers, analogous to linear algebra which uses the two operations of multiplication and arithmetic sum. He terms this matrix theory *minimax matrix theory.* The development of the theory is performed in the abstract, with an eye towards applications for matrices with values in the set $\mathbf{R}_{\pm\infty}$. The importance of Cuninghame-Green's work to the image algebra is that not only is the minimax matrix algebra embedded in the image algebra for the set $\mathbf{R}_{\pm\infty}$, but also for the set $\mathbf{R}^{+}_{\pm\infty}$. The set $(\mathbf{R}^{+}, \vee, \wedge, *)$ is an l-group also. An image algebra transform using either ⊠ or ⊘ can thus be viewed as a matrix transform in the minimax algebra for the respective case of $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^{+}_{\pm\infty}$. This completes the mathematical identification of the three main subalgebras in the image algebra. The linear transforms were classified by Gader [32] who showed that linear algebra is embedded into image algebra. As a result of each embedding above, the full power of the respective mathematical theory can be applied to solving problems in image processing, as long as the image processing problem can be formulated using image algebra operations of $\oplus$, ⊠, or ⊘. Since it has been formally proven that the image algebra can represent all image-to-image transforms (see section 1.1), the embeddings are very useful to have.

The rest of this section is devoted to introducing the basic notions of the minimax algebra structure and properties.

## 1.2.1. Basic Definitions and Notation

Let **F** be a semi-lattice ordered semi-group with semi-lattice operation $\vee$ and semi-group operation $\times$. Thus, **F** satisfies

$$x \vee ( y \vee z ) = ( x \vee y ) \vee z \qquad A_1$$

$$x \vee y = y \vee x \qquad A_2$$

$$x \vee x = x \qquad A_3$$

as it is a semi-lattice, as well as

$$x \times ( y \times z ) = ( x \times y ) \times z \qquad A_4$$

as it has an associative group operation $\times$, and

$$x \times ( y \vee z ) = ( x \times y ) \vee ( x \times z ) \qquad A_5$$

$$( y \vee z ) \times x = ( y \times x ) \vee ( z \times x ) \qquad A_6$$

as it is an ordered semi-group. We call this structure a *belt*, in the vein of rings. The operation $\vee$ is called an *addition*, and the operation $\times$ a *multiplication*. We shall also call a semi-lattice an *s-lattice*.

Suppose the belt **F** also satisfies the *dual* to axioms $A_1$ through $A_6$, where $\times'$ is another binary group multiplication:

$$x \wedge ( y \wedge z ) = ( x \wedge y ) \wedge z \qquad A'_1$$

$$x \wedge y = y \wedge x \qquad A'_2$$

$$x \wedge x = x \qquad A'_3$$

$$x \times' ( y \times' z ) = ( x \times' y ) \times' z \qquad A'_4$$

$$x \times' ( y \wedge z ) = ( x \times' y ) \wedge ( x \times' z ) \qquad A'_5$$

$$( y \wedge z ) \times' x = ( y \times' x ) \wedge ( z \times' x ). \qquad A'_6$$

Here, $\times'$ is called a *dual multiplication*, and $\wedge$ is called a *dual addition*. The (group) multiplication or dual multiplication is not assumed to be commutative.

If in addition to the above 12 axioms, $\mathbf{F}$ satisfies the following axiom,

$$x \vee (y \wedge x) = x \wedge (y \vee x) = x,$$

then $\mathbf{F}$ is a *belt with duality.* If the multiplication $\times$ and dual multiplication $\times'$ coincide, then we call the multiplication *self-dual.* A belt with duality and self-dual multiplication corresponds to a *lattice-ordered semi-group*, or l-semi-group, in lattice theory.

Let $(\mathbf{F}_1, \vee)$ and $(\mathbf{F}_2, \vee)$ be two s-lattices. A function f: $\mathbf{F}_1 \rightarrow \mathbf{F}_2$ is an *s-lattice homomorphism* if

$$f(x \vee y) = f(x) \vee f(y),$$

for all x,y $\in \mathbf{F}$. If $\mathbf{F}_1$ and $\mathbf{F}_2$ are belts and f: $\mathbf{F}_1 \rightarrow \mathbf{F}_2$ is an s-lattice homomorphism, then if f also satisfies

$$f(x \times y) = f(x) \times f(y)$$

for all x,y $\in \mathbf{F}$, then we say that f is a *belt homomorphism.* The following is an example of a belt *isomorphism.* Define f: $\mathbf{R} \rightarrow \mathbf{R}^+$ by

$$f(x) = e^x.$$

Then $f(x \vee y) = f(x) \vee f(y)$, and $f(x + y) = f(x) * f(y)$. It is trivial to show that f is a belt isomorphism.

The belts $\mathbf{R}$ and $\mathbf{R}^+$ are *commutative* belts, that is, the multiplication $\times$ commutes. Each also has an *identity element* under the multiplication, namely 0 for $\mathbf{R}$ and 1 for $\mathbf{R}^+$. Because they are groups, each element r $\in \mathbf{F}$ has a unique multiplicative inverse; we call such a belt a *division belt*, by analogy with division rings. A belt has a *null element* if there exists an element $\theta \in \mathbf{F}$ such that

$$\forall x \in \mathbf{F},\ x \vee \theta = x \text{ and } x \times \theta = \theta \times x = \theta.$$

The belts $(\mathbf{R}_{-\infty}, \vee, +)$ and $(\mathbf{R}^+_{-\infty}, \vee, +)$ each have the element $-\infty$ as its null element.

A division belt with distinct operations $\times$ and $\vee$ and with duality corresponds to a *lattice-ordered group*, or l-group. In fact, if $(\mathbf{F}, \vee, \times)$ is a belt with distinct operations $\vee$ and $\times$, then by defining

$$x \wedge y = ( x^{-1} \vee y^{-1} )^{-1} , \forall x,y \in \mathbf{F} \tag{1-2}$$

we have introduced a second (dual) s-lattice operation $\wedge$ such that $(\mathbf{F},\vee,\wedge)$ becomes a (distributive) lattice [35]. In our terms, the division belt $\mathbf{F}$ acquires a duality with a self-dual multiplication. Our main interest will be for the l-groups $(\mathbf{F}, \vee, \times, \wedge, \times') = (\mathbf{R}, \vee, +, \wedge, +)$ and $(\mathbf{R}^+, \vee, *, \wedge, *)$, $*$ representing real multiplication. From the above discussion, it follows that $(\mathbf{R}, \vee, +, \wedge, +)$ and $(\mathbf{R}^+, \vee, *, \wedge, *)$ are isomorphic as l-groups.

An arbitrary l-group $\mathbf{F}$ having two distinct binary operations $\vee$ and $\times$ can be extended in the following way. We adjoin the elements $+\infty$ and $-\infty$ to the set $\mathbf{F}$ and denote this new set by $\mathbf{F}_{\pm\infty}$, where $-\infty < x < +\infty \; \forall x \in \mathbf{F}$. We define a multiplication and a dual multiplication in $\mathbf{F}_{\pm\infty}$ by: if x,y $\in \mathbf{F}$, then $x \times y$ is already defined. Otherwise,

$$x \times -\infty = -\infty \times x = -\infty, \; x \in \mathbf{F} \cup \{-\infty\}$$

$$x \times +\infty = +\infty \times x = +\infty, \; x \in \mathbf{F} \cup \{+\infty\}$$

$$x \times' -\infty = -\infty \times' x = -\infty, \; x \in \mathbf{F} \cup \{-\infty\}$$

$$x \times' +\infty = +\infty \times' x = +\infty, \; x \in \mathbf{F} \cup \{+\infty\}.$$

$$-\infty \times +\infty = +\infty \times -\infty = -\infty$$

$$-\infty \times' +\infty = +\infty \times' -\infty = +\infty$$

The element $-\infty$ acts as a null element in the entire system $(\mathbf{F}_{\pm\infty}, \vee, \times)$ and the element $+\infty$ acts as a null element in the entire system $(\mathbf{F}_{\pm\infty}, \wedge, \times')$. However, the multiplications $\times$ and $\times'$ are asymmetric between the elements $-\infty$ and $+\infty$. The elements in $\mathbf{F}$ are called the *finite elements*.

We call such a system $(\mathbf{F}_{\pm\infty},\vee,\times,\wedge,\times')$ a *bounded l-group,* and $\mathbf{F}$ is called the *group* of the bounded l-group $\mathbf{F}_{\pm\infty}$.

The two bounded l-groups $(\mathbf{R}_{\pm\infty},\vee,+,\wedge,+')$ and $(\mathbf{R}^{+}_{\pm\infty},\vee,*,\wedge,*')$ will be our main concern. Another bounded l-group of interest is the *3-element* bounded l-group with group $\phi$, denoted by $\mathbf{F}_3$. Note that the boolean algebra $(\{-\infty,\phi\},\vee,\wedge)$ is embedded in $\mathbf{F}_3$, with OR $=\vee$ (maximum), AND $=\wedge$ (minimum), FALSE $=-\infty$, and TRUE $=\phi$. It is simple to check that the familiar truth tables hold.

Let $(\mathbf{F},\vee,\times)$ be a belt, and let $(\mathbf{T},\vee)$ be an s-lattice. Suppose we have a right multiplication of elements of $\mathbf{T}$ by elements of $\mathbf{F}$:

$$x\times\lambda\in\mathbf{T}\quad\forall \text{ pairs } x,\lambda,\ x\in\mathbf{T},\ \lambda\in\mathbf{F}.$$

We call $(\mathbf{T},\vee)$ a *right s-lattice space over* $(\mathbf{F},\vee,\times)$, or just say $\mathbf{T}$ *is a space over* $\mathbf{F}$ if the following axioms are satisfied for all $x,y\in\mathbf{T}$ and for all $\lambda,\mu\in\mathbf{F}$:

$$(\mathbf{T},\vee) \text{ is an s--lattice}$$

$$(x\times\lambda)\times\mu=x\times(\lambda\times\mu)$$

$$(x\vee y)\times\lambda=(x\times\lambda)\vee(y\times\lambda)$$

$$x\times(\lambda\vee\mu)=(x\times\lambda)\vee(x\times\mu)$$

and if $\mathbf{F}$ has an identity element $\phi$,

$$x\times\phi=x.$$

Such spaces play the role of vector spaces in the minimax theory. If $\mathbf{T}$ and $\mathbf{F}$ are known, then we shall simply say that $\mathbf{T}$ is a *space.*

A *subspace* is a subset of a space which is itself a space over the belt $\mathbf{F}$.

Let $(\mathbf{S},\vee)$, $(\mathbf{T},\vee)$ be given spaces over a belt $(\mathbf{F},\vee,\times)$. An s-lattice homomorphism

g: $(\mathbf{S}, \vee) \rightarrow (\mathbf{T}, \vee)$ is called *right linear* (*over* $\mathbf{F}$) if

$$g(x \times \lambda) = g(x) \times \lambda \quad \forall x \in \mathbf{S}, \forall \lambda \in \mathbf{F}.$$

We denote the set of all right-linear homomorphisms from $\mathbf{S}$ to $\mathbf{T}$ over $\mathbf{F}$ by $\mathrm{Hom}_{\mathbf{F}}(\mathbf{S}, \mathbf{T})$. That is,

$\mathrm{Hom}_{\mathbf{F}}(\mathbf{S}, \mathbf{T}) = \{g : \mathbf{S} \rightarrow \mathbf{T}$ is a homomorphism and $g(x \times \lambda) = g(x) \times \lambda \; \forall x \in \mathbf{S}, \forall \lambda \in \mathbf{F}\}$.

Let $(\mathbf{F}, \vee, \times)$ be a belt and $(\mathbf{T}, \vee)$ be an s-lattice, and suppose we have defined a left multiplication of elements of $\mathbf{T}$ by elements of $\mathbf{F}$:

$$\lambda \times x \in \mathbf{T} \quad \forall \text{ pairs } x, \lambda,\ x \in \mathbf{T},\ \lambda \in \mathbf{F}.$$

The left variants of the above five axioms are easily stated. We define a system satisfying those left axioms a *left* space over $\mathbf{F}$. This allows us to define a two-sided space. A *two-sided space* is a triple $(\mathbf{L}, \mathbf{T}, \mathbf{R})$ such that

$\mathbf{L}$ is a belt and $\mathbf{T}$ is a left space over $\mathbf{L}$.

$\mathbf{R}$ is a belt and $\mathbf{T}$ is a right space over $\mathbf{R}$.

$$\forall \lambda \in \mathbf{L}, \forall x \in \mathbf{T} \text{ and } \forall \mu \in \mathbf{R} : \lambda \times (x \times \mu) = (\lambda \times x) \times \mu.$$

Let $(\mathbf{F}, \vee, \times)$ be a belt. An important class of spaces over $\mathbf{F}$ is the class of function spaces. Here, the s-lattice $(\mathbf{T}, \vee)$ is $(\mathbf{F}^{U}, \vee)$. Such spaces are naturally two-sided. We shall only be interested in the case where $|U| = n \in \mathbf{Z}^{+}$. A space $(\mathbf{T}, \vee)$ is of form $(\mathbf{F}^{n}, \vee)$, and hence our spaces $\mathbf{F}^{n}$ are spaces of n-tuples.

When discussing conjugacy in linear operator theory, two approaches are commonly used. One defines the conjugate of a given space $\mathbf{S}$ as a special set $\mathbf{S}^{*}$ of linear, scalar-valued functions defined on $\mathbf{S}$. The other involves defining an *involution* taking $x \in \mathbf{S}$ to $x^{*} \in \mathbf{S}^{*}$ which satisfy certain axioms. (Recall a function f is an involution if $f^{-1}(f(x)) = x$.) The situation is slightly more complicated in the case of lattice transforms.

Let $(\mathbf{S},\vee,\times)$ and $(\mathbf{T},\wedge,\times')$ be given belts. We say that $(\mathbf{T},\wedge,\times')$ is *conjugate* to $(\mathbf{S},\vee,\times)$ if there is a function g: $\mathbf{S} \to \mathbf{T}$ such that

$$\text{g is bijective} \qquad C_1$$

$$\forall\, x,y \in \mathbf{S},\ g(x \vee y) = g(x) \wedge g(y) \qquad C_2$$

$$\forall\, x,y \in \mathbf{S},\ g(x \times y) = g(y) \times' g(x). \qquad C_3$$

In lattice theory, g is called a *dual isomorphism*. Note that conjugacy is a symmetric relation. If $(\mathbf{S},\vee,\wedge)$ is an s-lattice with duality satisfying the first two axioms, then we say that S is *self-conjugate*. If $(S,\vee,\times,\wedge,\times')$ a belt with duality, we say that $(S,\vee,\times,\wedge,\times')$ is *self-conjugate* if $(\mathbf{S},\wedge,\times')$ is conjugate to $(\mathbf{S},\vee,\times)$.

In particular, every division belt is self-conjugate under the bijection $x^* = x^{-1}$, and every bounded l-group is self-conjugate under the bijection $(-\infty)^* = +\infty$, $(+\infty)^* = -\infty$, and $x^* = x^{-1}$ if x is finite.

## 1.2.2. Matrix Algebra

We now present the extension of the belt operations to matrices. Let $(\mathbf{F},\vee,\times)$ be a belt. Let $\mathcal{M}_{mn}$ be the set of all m x n matrices with values in the set $\mathbf{F}$, and let $\mathbf{s} = (s_{ij})$, $\mathbf{t} = (t_{ij}) \in \mathcal{M}_{mn}$. Then we define

$$(s_{ij}) \vee (t_{ij}) \equiv (s_{ij} \vee t_{ij})$$

and for $(s_{ij}) \in \mathcal{M}_{mh}$, $(t_{jk}) \in \mathcal{M}_{hn}$, we have

$$(s_{ij}) \times (t_{jk}) \equiv \Big(\bigvee_{j=1}^{h} [s_{ij} \times t_{jk}]\Big) \in \mathcal{M}_{mn}.$$

Suppose $\mathbf{s} \in \mathcal{M}_{mn}$ and $\mathbf{t} \in \mathcal{M}_{hq}$. We say that $\mathbf{s}$ and $\mathbf{t}$ are *conformable for addition* whenever both m = h and n = q, and *conformable for multiplication* whenever n = h. For the remainder of this presentation, we use the notation $\mathbf{F}^n$ and $\mathcal{M}_{mn}$, as defined above. Also, we call an n-tuple or a matrix *finite* if all its elements are finite, i.e. not equal to either $+\infty$ or $-\infty$.

If $(\mathbf{F}, \vee, \times, \wedge, \times')$ is a belt with duality, then we say that a space $(\mathbf{T}, \vee)$ over $\mathbf{F}$ has a *duality* if

a dual addition $\wedge$ is defined where $(\mathbf{T}, \vee, \wedge)$ is an s–lattice with duality;

$(\mathbf{T}, \wedge)$ is a space over the belt $(\mathbf{F}, \wedge, \times')$.

We also have a dual matrix addition and dual multiplication defined for matrices over a belt with duality.

$$(s_{ij}) \wedge (t_{ij}) \equiv (s_{ij} \wedge t_{ij})$$

and for $(s_{ij}) \in \mathcal{M}_{mh}$, $(t_{jk}) \in \mathcal{M}_{hn}$, we have

$$(s_{ij}) \times' (t_{jk}) \equiv \left( \bigwedge_{j=1}^{h} [s_{ij} \times' t_{jk}] \right) \in \mathcal{M}_{mn}$$

with the expressions *conformable for dual addition* $\wedge$ and *conformable for dual multiplication* $\times'$ used in the obvious way.

Let $(\mathbf{F}, \vee, \times)$ be a belt and let $\mathcal{M}_{pq}$ denote the set of p x q matrices with values in $\mathbf{F}$. The following are some basic properties that are proven in [38].

(1) $(\mathcal{M}_{mn}, \vee)$ is an s-lattice and $(\mathcal{M}_{np}, \vee)$ is a function space over $(\mathbf{F}, \vee, \times)$;

(2) $(\mathcal{M}_{nn}, \vee, \times)$ is a belt;

(3) $(\mathcal{M}_{np}, \vee)$ is a left space over the belt $(\mathcal{M}_{nn}, \vee, \times)$;

(4) $\mathcal{M}_{np}$ is a right space over the belt $\mathbf{F}$;

(5) Scalar multiplication of a matrix $\mathbf{s}$ by an element $\lambda \in \mathbf{F}$ is defined by

$$(s_{ij}) \times \lambda \equiv (s_{ij} \times \lambda)$$

$$\lambda \times (s_{ij}) \equiv (\lambda \times s_{ij})$$

for all $(s_{ij}) \in \mathcal{M}_{np}$, $\lambda \in \mathbf{F}$;

(6) For all $\mathbf{s} \in \mathcal{M}_{mn}$, $\mathbf{t}, \mathbf{u} \in \mathcal{M}_{np}$, $\lambda \in \mathbf{F}$,

$$\mathbf{s} \times (\mathbf{t} \vee \mathbf{u}) = (\mathbf{s} \times \mathbf{t}) \vee (\mathbf{s} \times \mathbf{u})$$
$$\mathbf{s} \times (\mathbf{t} \times \lambda) = (\mathbf{s} \times \mathbf{t}) \times \lambda.$$

Since the s-lattice $(\mathcal{M}_{n1}, \vee)$ is isomorphic to the s-lattice $\mathbf{F}^n$, we have $\mathbf{F}^n$ is a function space over $\mathbf{F}$ as well as a space over $\mathcal{M}_{nn}$. This mimics the classical role of matrices as linear transformations of spaces of n-tuples!

Two important matrices in our present setting are the identity matrix and the null matrix. Suppose the belt $\mathbf{F}$ has identity and null elements $\phi$ and $-\infty$ respectively. We define the *identity matrix* $\mathbf{e} \in \mathcal{M}_{nn}$ by

$$\mathbf{e} = \begin{bmatrix} \phi & . & . & . & . \\ . & \phi & . & -\infty & . \\ . & . & . & . & . \\ . & -\infty & . & . & . \\ . & . & . & . & \phi \end{bmatrix}$$

and the *null matrix* $\Phi \in \mathcal{M}_{mn}$ by

$$\Phi = \begin{bmatrix} -\infty & . & . & . & . \\ . & -\infty & . & -\infty & . \\ . & . & . & . & . \\ . & -\infty & . & . & . \\ . & . & . & . & -\infty \end{bmatrix}.$$

Thus we have $\forall\, \mathbf{s} \in \mathcal{M}_{nn}$ and for $\Phi \in \mathcal{M}_{nn}$,

$$\mathbf{e} \times \mathbf{s} = \mathbf{s} \times \mathbf{e} = \mathbf{s}$$

$$\mathbf{s} \vee \Phi = \mathbf{s}$$

$$\mathbf{s} \times \Phi = \Phi \times \mathbf{s} = \Phi.$$

In the bounded l-group $\mathbf{R}_{\pm\infty}$ we have

$$\mathbf{e} = \begin{bmatrix} 0 & . & . & . & . \\ . & 0 & . & -\infty & . \\ . & . & . & . & . \\ . & -\infty & . & . & . \\ . & . & . & . & 0 \end{bmatrix}$$

and in $\mathbf{R}_{\pm\infty}^{+}$ we have

$$\mathbf{e} = \begin{bmatrix} 1 & . & . & . & . \\ . & 1 & . & -\infty & . \\ . & . & . & . & . \\ . & -\infty & . & . & . \\ . & . & . & . & 1 \end{bmatrix}.$$

Conjugacy extends to matrices if the underlying value set is itself a self-conjugate belt. This is stated in the next proposition.

**Proposition 1.1** [38]. *If* $(\mathbf{F}, \vee, \times, \wedge, \times')$ *is a self-conjugate belt, then* $(\mathcal{M}_{nn}, \vee, \times, \wedge, \times')$ *is a self-conjugate belt.*

In linear algebra, we characterize linear transformations of vector spaces entirely in terms of matrices. Are we able to do a similar classification here? The following results give necessary and sufficient conditions for this to be the case.

**Theorem 1.2** [38]. *Let* $\mathbf{F}$ *be a belt which has an identity element* $\phi$ *with respect to* $\times$ *and a null element* $\theta$ *with respect to* $-\infty$. *Then for all integers* m,n $\geq 1$, $\mathcal{M}_{mn}$ *is isomorphic to* $\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^n, \mathbf{F}^m)$.

**Corollary 1.3** [38]. *Let* $\mathbf{F}$ *be a belt, and let* $n > 1$ *be a given integer. Then a necessary and sufficient condition that* $\mathcal{M}_{mn}$ *be isomorphic to* $\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^n, \mathbf{F}^m)$ *for all integers* n,m $\geq 1$ *is that* $\mathbf{F}$ *have an identity element* $\phi$ *with respect to* $\times$ *and a null element* $\theta$ *with respect to* $\vee$.

We call a matrix $\mathbf{s} \in \mathcal{M}_{nm}$ a *lattice transform.*

Many of the results that were stated in Cuninghame's book can be viewed in in context of a dual lattice-ordered semi-group, which has been extensively researched [35]. However, we wish to study the structure from a different perspective. The extension of the belt operations to matrices allows us to view matrices as operators on spaces of n-tuples, in a way similar to vector-space transformations. These operators are non-linear due to the lattice

structure of the underlying set **F**. Thus, we may study this particular class of non-linear transforms in a mathematically rigorous setting, and, since an image can be viewed as a vector and a template as a matrix (as will be shown in Chapter 2), apply results from the minimax matrix theory directly to solve image processing problems. For example, decomposition of matrices corresponds to decomposition of templates. This particular application is discussed in Chapter 5.

# CHAPTER 2
# THE ISOMORPHISM

In his Ph.D. dissertation, P. Gader showed that linear algebra can be embedded into the image algebra [32]. One very powerful implication of this is that all the tools of linear algebra are directly applicable to solving problems in image processing whenever the image algebra operation $\oplus$ is involved. We now show an embedding of the minimax algebra into image algebra for the two cases where the belts are $\mathbf{R}$ and $\mathbf{R}^+$. We employ the same functions $\Psi$ and $\nu$ as used by Gader in his dissertation.

Let $\mathbf{X}$ and $\mathbf{Y}$ be finite arrays, with $|\mathbf{X}| = \mathrm{m}$ and $|\mathbf{Y}| = \mathrm{n}$. Assume the points of $\mathbf{X}$ are labelled lexicographically $\mathbf{x}_1, \mathbf{x}_2, ..., \mathbf{x}_m$. Assume a similar labelling for $\mathbf{Y}$: $\mathbf{Y} = \{\mathbf{y}_1, \mathbf{y}_2, ..., \mathbf{y}_n\}$. Let $\mathbf{R}_{\pm\infty}$ have its usual meaning. Let $\mathbf{R}^m_{\pm\infty} = \{(x_1, x_2, ..., x_m) : x_i \in \mathbf{R}_{\pm\infty}\}$. That is, $\mathbf{R}^m_{\pm\infty}$ is the set of row vectors of m-tuples with values in $\mathbf{R}_{\pm\infty}$. Let $\mathbf{a} \in \mathbf{R}^{\mathbf{X}}_{\pm\infty}$, $\mathcal{M}_{mn}$ denote the set of m x n matrices with values in $\mathbf{R}_{\pm\infty}$, and define $\nu : \mathbf{R}^{\mathbf{X}}_{\pm\infty} \rightarrow \mathbf{R}^m_{\pm\infty}$ by

$$\nu(\mathbf{a}) = (\mathbf{a}(\mathbf{x}_1), ..., \mathbf{a}(\mathbf{x}_m)).$$

Define $\Psi : (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}} \rightarrow \mathcal{M}_{mn}$ by

$$\Psi(\mathbf{t}) = \mathrm{M}_\mathbf{t} = (\mathrm{p}_{ij}), \text{ where } \mathrm{p}_{ij} = \mathbf{t}_{\mathbf{y}_j}(\mathbf{x}_i).$$

Note that the j-th column of $\mathrm{M}_\mathbf{t}$ is simply $(\nu(\mathbf{t}_j))'$, the prime denoting transpose.

In the following lemmas, we assume that $|\mathbf{X}| = \mathrm{m}$, $|\mathbf{Y}| = \mathrm{n}$, and $|\mathbf{W}| = l$. We claim the following:

**Lemma 2.1.** $\nu(\mathbf{a} \boxtimes \mathbf{t}) = \nu(\mathbf{a}) \times \Psi(\mathbf{t})$, *for* $\mathbf{t} \in (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$, $\mathbf{a} \in \mathbf{R}^{\mathbf{X}}_{\pm\infty}$.

**Lemma 2.2.** $\nu(\mathbf{a} \vee \mathbf{b}) = \nu(\mathbf{a}) \vee \nu(\mathbf{b})$, $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}_{\pm\infty}$, $\mathbf{F} \in \{\mathbf{R}, \mathbf{R}^+\}$.

**Lemma 2.3.** $\Psi(\mathbf{s} \boxtimes \mathbf{t}) = \Psi(\mathbf{s}) \times \Psi(\mathbf{t})$, *for* $\mathbf{s} \in (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{W}}$, $\mathbf{t} \in (\mathbf{R}^{\mathbf{W}}_{\pm\infty})^{\mathbf{Y}}$.

**Lemma 2.4.** $\Psi(\mathbf{s} \vee \mathbf{t}) = \Psi(\mathbf{s}) \vee \Psi(\mathbf{t})$, $\mathbf{s}, \mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$, $\mathbf{F} \in \{\mathbf{R}, \mathbf{R}^+\}$.

The proofs are given below.

**Proof to Lemma 2.1.**

We must show that

$$(\mathbf{a} \boxtimes \mathbf{t})(\mathbf{y}_k) = (\nu(\mathbf{a}) \times \Psi(\mathbf{t}))_k .$$

First note that $\nu(\mathbf{a} \boxtimes \mathbf{t})$ is a 1 x n row vector, as is $\nu(\mathbf{a}) \times \Psi(\mathbf{t})$. We have

$$(\mathbf{a} \boxtimes \mathbf{t})(\mathbf{y}_k) = \bigvee_{\mathbf{x} \in \mathbf{X}} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}) = \bigvee_{i=1}^{m} \mathbf{a}(\mathbf{x}_i) + \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_i) .$$

Also, $(\nu(\mathbf{a}) \times \Psi(\mathbf{t}))_k = \bigvee_{j=1}^{m} (\nu(\mathbf{a}))_j + (\Psi(\mathbf{t}))_{jk} = \bigvee_{j=1}^{m} \mathbf{a}(\mathbf{x}_j) + \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_j)$.

Q.E.D.

**Proof to Lemma 2.2.**

At location $\mathbf{x}_k$, the image $\mathbf{a} \vee \mathbf{b}$ has value $\mathbf{a}(\mathbf{x}_k) \vee \mathbf{b}(\mathbf{x}_k)$. At location k, the row vector $\nu(\mathbf{a}) \vee \nu(\mathbf{b})$ has value $(\nu(\mathbf{a}))_k \vee (\nu(\mathbf{b}))_k = \mathbf{a}(\mathbf{x}_k) \vee \mathbf{b}(\mathbf{x}_k)$.

Q.E.D.

**Proof to Lemma 2.3.**

Here, $\mathbf{s} \in (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{W}}$ and $\mathbf{t} \in (\mathbf{R}^{\mathbf{W}}_{\pm\infty})^{\mathbf{Y}}$ implies

$$\mathbf{s} \boxtimes \mathbf{t} = \mathbf{r} \in (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}},$$

and

$$\mathbf{r}_{\mathbf{y}_j}(\mathbf{x}_i) = \bigvee_{\mathbf{w} \in \mathbf{W}} \mathbf{t}_{\mathbf{y}_j}(\mathbf{w}) + \mathbf{s}_{\mathbf{w}}(\mathbf{x}_i) = \bigvee_{k=1}^{l} \mathbf{t}_{\mathbf{y}_j}(\mathbf{w}_k) + \mathbf{s}_{\mathbf{w}_k}(\mathbf{x}_i) .$$

Now, let $\Psi(\mathbf{s}) \times \Psi(\mathbf{t}) = \mathbf{u} \in \mathcal{M}_{mn}$. We have

$$u_{ij} = \bigvee_{k=1}^{l} (\Psi(\mathbf{s}))_{ik} + (\Psi(\mathbf{t}))_{kj} = \bigvee_{k=1}^{l} \mathbf{s}_{w_k}(\mathbf{x}_i) + \mathbf{t}_{y_j}(\mathbf{w}_k) = \bigvee_{k=1}^{l} \mathbf{t}_{y_j}(\mathbf{w}_k) + \mathbf{s}_{w_k}(\mathbf{x}_i).$$

Q.E.D.

**Proof to Lemma 2.4.**

Here, $\mathbf{s}$, $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$. Then

$$(\mathbf{s} \vee \mathbf{t})_{y_j}(\mathbf{x}_i) = \mathbf{s}_{y_j}(\mathbf{x}_i) \vee \mathbf{t}_{y_j}(\mathbf{x}_i),$$

while

$$(\Psi(\mathbf{s}) \vee \Psi(\mathbf{t}))_{ij} = (\Psi(\mathbf{s}))_{ij} \vee (\Psi(\mathbf{t}))_{ij} = \mathbf{s}_{y_j}(\mathbf{x}_i) \vee \mathbf{t}_{y_j}(\mathbf{x}_i).$$

Q.E.D.

In order to prove the isomorphism theorem, we will use the following lemma.

**Lemma 2.5.** $\Psi(\mathbf{t}^*) = (\Psi(\mathbf{t}))^*$, $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$, *where* $\mathbf{F}$ *denotes either* $\mathbf{R}$ *or* $\mathbf{R}^+$. *In this particular instance we let* $\mathbf{t}^*$ *denote the conjugate template of* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$.

**Proof:** Let $\mathbf{s} = \mathbf{t}^*$. Then $\mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{Y}})^{\mathbf{X}}$, and

$$\Psi(\mathbf{t}^*) = \Psi(\mathbf{s}) = M_{\mathbf{s}} = (p_{ij}), \text{ where } p_{ij} = \mathbf{s}_{x_j}(\mathbf{y}_i) = (\mathbf{t}^*)_{x_j}(\mathbf{y}_i) = [(\mathbf{t}_{y_i}(\mathbf{x}_j)]^*,$$

while

$$\Psi(\mathbf{t}) = M_{\mathbf{t}} = (q_{ij}), \text{ where } q_{ij} = \mathbf{t}_{y_j}(\mathbf{x}_i).$$

Obviously,

$$p_{ij} = [\mathbf{t}_{y_i}(\mathbf{x}_j)]^* = [q_{ji}]^*.$$

Thus,

$$M_{\mathbf{s}} = (p_{ij}) = ([q_{ji}]^*) = (q_{ij})^* = (M_{\mathbf{t}})^*,$$

and we have $\Psi(\mathbf{t}^*) = M_{\mathbf{s}} = (M_{\mathbf{t}})^* = (\Psi(\mathbf{t}))^*$.

Q.E.D.

The following theorem, along with Lemmas 2.1 through 2.4, show how the embedding of the minimax algebra into the image algebra is accomplished.

**Theorem 2.6.** *For a finite array* $\mathbf{X}$*, with* $|\mathbf{X}| = \mathrm{m}$,

$$\{\,\mathbf{R}_{\pm\infty}^{\mathbf{X}}, \vee, \wedge; (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}},\ \boxed{\vee}\ , \vee,\ \boxed{\wedge}\ , \wedge\,;\boxed{\vee}\ ,\boxed{\wedge}\,\}\ \textit{is isomorphic to}$$

$$\{\mathbf{R}_{\pm\infty}^{\mathrm{m}}, \vee, \wedge; \mathcal{M}_{\mathrm{mm}}, \times\ , \vee, \times', \wedge; \times\ , \times'\},$$

*where* $\mathcal{M}_{\mathrm{mm}}$ *is the set of all* m x m *matrices with entries in the bounded l-group* $\mathbf{R}_{\pm\infty}$.

**Proof:** By Lemma 2.1, $\nu$ preserves image-template multiplication, and by Lemma 2.2, $\nu$ preserves the image-image pointwise maximum operation. By Lemmas 2.3 and 2.4, for $\mathbf{X} = \mathbf{Y} = \mathbf{W}$, $\Psi$ preserves the operations of $\boxed{\vee}$ and $\vee$ between templates. Let $\mathbf{l} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ denote the identity template defined by

$$\mathbf{l}_{\mathbf{y}}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{y} = \mathbf{x} \\ -\infty & \text{otherwise} \end{cases}.$$

It is trivial to show that $\Psi(\mathbf{l}) = \mathbf{e} \in \mathcal{M}_{\mathrm{mm}}$, the identity matrix in $\mathcal{M}_{\mathrm{mm}}$.

We now show that the operations of $\boxed{\wedge}$ and $\wedge$ are also preserved under $\Psi$. It is not difficult to show that $\Psi(\mathbf{s} \wedge \mathbf{t}) = \Psi(\mathbf{s}) \wedge \Psi(\mathbf{t})$. Let $\mathbf{r} = \mathbf{s} \wedge \mathbf{t}$. Then $\Psi(\mathbf{r}) = \mathrm{M}_{\mathbf{r}} = (\mathrm{m}_{ij}) = (\mathbf{r}_{\mathbf{y}_j}(\mathbf{x}_i))$, where $\mathbf{r}_{\mathbf{y}_j}(\mathbf{x}_i) = \mathbf{s}_{\mathbf{y}_j}(\mathbf{x}_i) \wedge \mathbf{t}_{\mathbf{y}_j}(\mathbf{x}_i)$. Thus,

$$\Psi(\mathbf{s}) \wedge \Psi(\mathbf{t}) = \mathrm{M}_{\mathbf{s}} \wedge \mathrm{M}_{\mathbf{t}} = (\mathbf{s}_{\mathbf{y}_j}(\mathbf{x}_i)) \wedge (\mathbf{t}_{\mathbf{y}_j}(\mathbf{x}_i)) = (\mathbf{s}_{\mathbf{y}_j}(\mathbf{x}_i) \wedge \mathbf{t}_{\mathbf{y}_j}(\mathbf{x}_i)) = (\mathbf{r}_{\mathbf{y}_j}(\mathbf{x}_i)).$$

By definition, $\mathbf{s}\ \boxed{\wedge}\ \mathbf{t} = (\mathbf{t}^*\ \boxed{\vee}\ \mathbf{s}^*)^*$, and, using Lemma 2.5 with $\mathbf{F} = \mathbf{R}$, Lemma 2.3, and property $\mathrm{C}_3$, we have

$$\Psi(\mathbf{s}\ \boxed{\wedge}\ \mathbf{t}) = \Psi((\mathbf{t}^*\ \boxed{\vee}\ \mathbf{s}^*)^*) = [\Psi(\mathbf{t}^*\ \boxed{\vee}\ \mathbf{s}^*)]^* = (\Psi(\mathbf{t}^*) \times \Psi(\mathbf{s}^*))^*$$
$$= (\Psi(\mathbf{s}^*))^* \times' (\Psi(\mathbf{t}^*))^* = \Psi(\mathbf{s}) \times' \Psi(\mathbf{t}).$$

Thus, $\Psi(\mathbf{s}\ \boxed{\wedge}\ \mathbf{t}) = \Psi(\mathbf{s}) \times' \Psi(\mathbf{t})$.

It is straightforward to see that $\nu$ is on-to-one and onto $\mathbf{R}_{\pm\infty}^{\mathrm{m}}$. To show that $\Psi$ is one-one and onto $\mathcal{M}_{\mathrm{mn}}$, let $\mathbf{s}, \mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ and suppose that $\Psi(\mathbf{s}) = \Psi(\mathbf{t})$. Then

$$(\Psi(\mathbf{s}))_{ij} = (M_s)_{ij} = \mathbf{s}_{y_j}(\mathbf{x}_i) = \mathbf{t}_{y_j}(\mathbf{x}_i) = (M_t)_{ij} = (\Psi(\mathbf{t}))_{ij}, \text{ and, thus,}$$

$$\mathbf{s}_{y_j}(\mathbf{x}_i) = \mathbf{t}_{y_j}(\mathbf{x}_i) \text{ for all } j = 1,\dots,n, \text{ and for all } i = 1,\dots,m.$$

So $\Psi$ is one-to-one as $\mathbf{s} = \mathbf{t}$. Let $M = (m_{ij}) \in \mathcal{M}_{mn}$. Define $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ by $\mathbf{t}_{y_j}(\mathbf{x}_i) = m_{ij}$. Then $\Psi(\mathbf{t}) = M$. Setting $m = n$, we see that $\Psi$ is one-one and onto $\mathcal{M}_{mm}$.

Q.E.D.

Thus, the minimax algebra with the bounded l-group $\mathbf{R}_{\pm\infty}$ is embedded into image algebra, by the functions $\Psi^{-1}$ and $\nu^{-1}$. As the bounded l-group $\mathbf{R}_{\pm\infty}^{+}$ is isomorphic to the bounded l-group $\mathbf{R}_{\pm\infty}$, the minimax algebra with the bounded l-group $\mathbf{R}_{\pm\infty}^{+}$ is also embedded into the image algebra. In this case, the matrix operation $\times$ corresponds to the image algebra operation $\circledR$. The isomorphism result is stated in Theorem 2.9.

Let $\mathbf{X}$ and $\mathbf{Y}$ be finite arrays as before. Let $\mathbf{R}_{\pm\infty}^{+}$ have its usual meaning, $\mathbf{a} \in (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}}$, $\mathcal{M}_{mn}$ denote the set of m x n matrices with values in $\mathbf{R}_{\pm\infty}^{+}$, and let $(\mathbf{R}_{\pm\infty}^{+})^m = \{(x_1,x_2,\dots,x_m) : x_i \in \mathbf{R}_{\pm\infty}^{+}\}$. Define $\nu : (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}} \rightarrow (\mathbf{R}_{\pm\infty}^{+})^m$ in the usual way by

$$\nu(\mathbf{a}) = (\mathbf{a}(\mathbf{x}_1),\dots,\mathbf{a}(\mathbf{x}_m)).$$

Define $\Psi : ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}} \rightarrow \mathcal{M}_{mn}$ as before by

$$\Psi(\mathbf{t}) = M_t = (p_{ij}), \text{ where } p_{ij} = \mathbf{t}_{y_j}(\mathbf{x}_i).$$

In the following lemmas, we assume that $|\mathbf{X}| = m$, $|\mathbf{Y}| = n$, and $|\mathbf{W}| = l$. We claim the following, for $\mathbf{a}$, $\mathbf{b} \in (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}}$:

**Lemma 2.7.** $\nu(\mathbf{a} \circledR \mathbf{t}) = \nu(\mathbf{a}) \times \Psi(\mathbf{t})$, *for* $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}}$.

**Lemma 2.8.** $\Psi(\mathbf{s} \circledR \mathbf{t}) = \Psi(\mathbf{s}) \times \Psi(\mathbf{t})$, *for* $\mathbf{s} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{W}}$, $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{W}})^{\mathbf{Y}}$.

**Proof to Lemma 2.7.**

We must show that

$$(\mathbf{a} \circledvee \mathbf{t})(\mathbf{y}_k) = (\nu(\mathbf{a}) \times \Psi(\mathbf{t}))_k .$$

We have

$$(\mathbf{a} \circledvee \mathbf{t})(\mathbf{y}_k) = \bigvee_{\mathbf{x}\in\mathbf{X}} \mathbf{a}(\mathbf{x}) * \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}) = \bigvee_{i=1}^{m} \mathbf{a}(\mathbf{x}_i) * \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_i) .$$

Also, $(\nu(\mathbf{a}) \times \Psi(\mathbf{t}))_k = \bigvee_{j=1}^{m} (\nu(\mathbf{a}))_j * (\Psi(\mathbf{t}))_{jk} = \bigvee_{j=1}^{m} \mathbf{a}(\mathbf{x}_j) * \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_j).$

Q.E.D.

**Proof to Lemma 2.8.**

Here, $\mathbf{s} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{W}}$ and $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{W}})^{\mathbf{Y}}$ implies

$$\mathbf{s} \circledvee \mathbf{t} = \mathbf{r} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}},$$

and

$$\mathbf{r}_{\mathbf{y}_j}(\mathbf{x}_i) = \bigvee_{\mathbf{w}\in\mathbf{W}} \mathbf{t}_{\mathbf{y}_j}(\mathbf{w}) * \mathbf{s}_{\mathbf{w}}(\mathbf{x}_i) = \bigvee_{k=1}^{l} \mathbf{t}_{\mathbf{y}_j}(\mathbf{w}_k) * \mathbf{s}_{\mathbf{w}_k}(\mathbf{x}_i) .$$

Now, let $\Psi(\mathbf{s}) \times \Psi(\mathbf{t}) = \mathbf{u} \in \mathcal{M}_{mn}$. We have

$$u_{ij} = \bigvee_{k=1}^{l} (\Psi(\mathbf{s}))_{ik} * (\Psi(\mathbf{t}))_{kj} = \bigvee_{k=1}^{l} \mathbf{s}_{\mathbf{w}_k}(\mathbf{x}_i) * \mathbf{t}_{\mathbf{y}_j}(\mathbf{w}_k) = \bigvee_{k=1}^{l} \mathbf{t}_{\mathbf{y}_j}(\mathbf{w}_k) * \mathbf{s}_{\mathbf{w}_k}(\mathbf{x}_i) .$$

Q.E.D.

**Theorem 2.9.** *For a finite array* $\mathbf{X}$, *with* $|\mathbf{X}| = m$,

$$\{(\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}}, \vee, \wedge; ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{X}}, \circledvee, \vee, \circledwedge, \wedge; \circledvee, \circledwedge\} \text{ is isomorphic to}$$

$$\{(\mathbf{R}_{\pm\infty}^{+})^{m}, \vee, \wedge; \mathcal{M}_{mm}, \times, \vee, \times', \wedge; \times, \times'\},$$

*where* $\mathcal{M}_{mm}$ *is the set of all* m x m *matrices with entries in the bounded l-group* $\mathbf{R}_{\pm\infty}^{+}$.

**Proof:** By Lemma 2.7, $\nu$ preserves image-template multiplication, and by Lemma 2.2, $\nu$ preserves the image-image pointwise maximum operation. By Lemmas 2.8 and 2.4,

for $\mathbf{X} = \mathbf{Y} = \mathbf{W}$, $\Psi$ preserves the operations of $\boxtimes$ and $\bigvee$ between templates. Let $\mathbf{l} \in ((\mathbf{R}^+_{\pm\infty})^{\mathbf{X}})^{\mathbf{X}}$ denote the identity template defined by

$$\mathbf{l}_{\mathbf{y}}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{y} = \mathbf{x} \\ -\infty & \text{otherwise} \end{cases}.$$

It is trivial to show that $\Psi(\mathbf{l}) = \mathbf{e} \in \mathcal{M}_{mm}$, the identity matrix in $\mathcal{M}_{mm}$ over $\mathbf{R}^+_{\pm\infty}$.

In Theorem 2.6, the proof that $\Psi(\mathbf{s} \wedge \mathbf{t}) = \Psi(\mathbf{s}) \wedge \Psi(\mathbf{t})$ was not dependent on the value set $\mathbf{R}_{\pm\infty}$, and hence is true also for templates $\mathbf{s}, \mathbf{t} \in ((\mathbf{R}^+_{\pm\infty})^{\mathbf{X}})^{\mathbf{Y}}$. We now show that the operation of $\bigotimes$ is also preserved under $\Psi$. By definition, $\mathbf{s} \bigotimes \mathbf{t} = \overline{(\bar{\mathbf{t}} \bigcirc\!\!\!\vee\, \bar{\mathbf{s}})}$, and, using Lemma 2.5 with $\mathbf{F} = \mathbf{R}^+$, Lemma 2.8, and property $C_3$, we have

$$\begin{aligned} \Psi(\mathbf{s} \bigotimes \mathbf{t}) &= \Psi(\overline{(\bar{\mathbf{t}} \bigcirc\!\!\!\vee\, \bar{\mathbf{s}})}) = [\Psi(\bar{\mathbf{t}} \bigcirc\!\!\!\vee\, \bar{\mathbf{s}})]^* = [\Psi(\bar{\mathbf{t}}) \times \Psi(\bar{\mathbf{s}})]^* \\ &= (\Psi(\bar{\mathbf{s}}))^* \times' (\Psi(\bar{\mathbf{t}}))^* = \Psi(\mathbf{s}) \times' \Psi(\mathbf{t}). \end{aligned}$$

Thus, $\Psi(\mathbf{s} \bigotimes \mathbf{t}) = \Psi(\mathbf{s}) \times' \Psi(\mathbf{t})$.

We use the fact that Theorem 2.6 showed $\Psi$ and $\nu$ are one-one and onto and also that $\mathbf{R}_{\pm\infty}$ and $\mathbf{R}^+_{\pm\infty}$ are isomorphic as bounded l-groups, and we are done.

Q.E.D.

We have shown that the minimax algebra with two different interpretations for the bounded l-group $\mathbf{F}_{\pm\infty}$ with group $\mathbf{F}$, namely $\mathbf{F} = \mathbf{R}$ and $\mathbf{F} = \mathbf{R}^+$, is embedded in the image algebra. Using the notation $\mathbf{R}^+_{\pm\infty}$ instead of $\mathbf{R}^{\geq 0}_{+\infty}$ allows the reader to regard the value sets $\mathbf{R}^+_{\pm\infty}$ and $\mathbf{R}_{\pm\infty}$ as basically the same (they are isomorphic as belts), without shifting gears from using 0 in one as the bottom element and $-\infty$ in the other. All minimax properties stated in Cuninghame-Green's book will be valid in the correct context of image algebra notation.

In using the minimax algebra results, we would like to point out that the the matrix-vector multiplication, multiplication of a matrix by a vector from the right, is used mostly throughout Cuninghame-Green's book. Left multiplication is mentioned at various places, and in fact, most left variants of the right multiplication results will hold. However, for the most part in our applications to image algebra, we will be using the right multiplication form in the development of our theory. The functions $\Psi$ and $\nu$ map the image algebra expression $\mathbf{a} \boxed{\vee} \mathbf{t} = \mathbf{b}$ to the matrix algebra expression $\nu(\mathbf{a}) \times \Psi(\mathbf{t}) = \nu(\mathbf{b})$, the left multiplication form which we have omitted in our presentation of Cuninghame's material. The following diagram in Figure 5 explains how we will be taking advantage of the minimax algebra results.

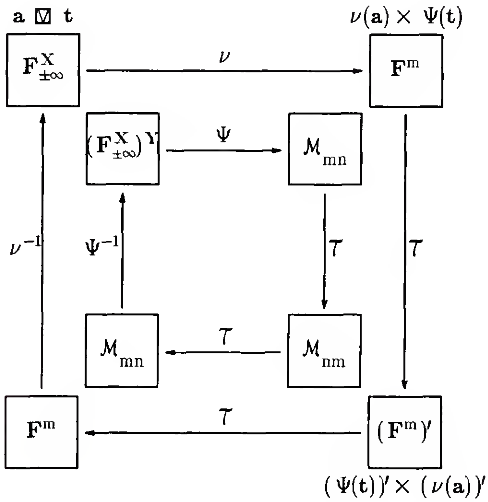



**Figure 5.** How the Transpose is used in Conjunction with the Isomorphism.

Let $\mathcal{T}$ denote the function that takes a matrix to its transpose as well as the function that

takes a template to its transpose. Thus, $\mathcal{T}: \mathcal{M}_{mn} \rightarrow \mathcal{M}_{nm}$ is defined by

$$\mathcal{T}(\mathbf{a}) = \mathbf{a}',$$

the prime denoting as usual the transpose of a matrix, and $\mathcal{T}: (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}} \rightarrow (\mathbf{F}^{\mathbf{Y}})^{\mathbf{X}}$ is defined by

$$\mathcal{T}(\mathbf{t}) = \mathbf{t}'.$$

Obviously, $\Psi(\mathcal{T}(\mathbf{t})) = \mathcal{T}(\Psi(\mathbf{t}))$. In a clockwise manner, the functions $\nu$ and $\Psi$ take the product $\nu(\mathbf{a} \boxtimes \mathbf{t})$ to $\nu(\mathbf{a}) \times \Psi(\mathbf{t})$, which is the matrix $\Psi(\mathbf{t})$ multiplied on the left by the row vector $\nu(\mathbf{a})$. Applying the transpose to $\nu(\mathbf{a}) \times \Psi(\mathbf{t})$, we get $\mathcal{T}[\nu(\mathbf{a}) \times \Psi(\mathbf{t})] = [\Psi(\mathbf{t})]' \times [\nu(\mathbf{a})]'$, which is the matrix $[\Psi(\mathbf{t})]' \in \mathcal{M}_{nm}$ multiplied on the right by the column vector $[\nu(\mathbf{a})]'$. We now use our minimax algebra theorems, where matrix-vector multiplication is the matrix multiplied on its right by a column vector. After getting the desired results, we continue on around the diagram clockwise, mapping back by the transpose $\mathcal{T}$ again and then by $\nu^{-1}$ or $\Psi^{-1}$. Formally, if $\mathbf{d}$ represents the column vector which is the result of applications of minimax algebra theorems to the initial column vector $(\Psi(\mathbf{t}))' \times (\nu(\mathbf{a}))'$, then $\nu^{-1}(\mathcal{T}(\mathbf{d}))$ will be an image. A similar situation holds for templates.

The minimax algebra results are stated in the usual matrix-vector multiplication order, and the isomorphisms $\Psi$ and $\nu$ are used along with the transpose $\mathcal{T}$ to apply the matrix results. When the word *isomorphism* is used in this context, it will mean the above functions $\Psi$ and $\nu$ explicitly (not with the transpose) unless otherwise stated, with images as row vectors and templates as matrices with images $\mathbf{t}_y$ as columns.

## PART II
## MINIMAX APPLICATIONS TO IMAGE ALGEBRA AND IMAGE PROCESSING

The objective of the chapters in Part II is to show how the minimax algebra can be used to extend basic matrix algebraic results in such a way as to have applications in image processing. The tool that makes the minimax algebra useful in image processing is the isomorphism between the image algebra and the minimax algebra. Before the research presented in this dissertation was conducted, the relationship between the image algebra and the minimax algebra had not been established. The power of the isomorphism is that it makes all results in the minimax algebra applicable to solving image processing problems, just as linear algebra results are applicable to solving image processing problems. For example, template decomposition is presently a very active area of research. The problem of mapping transforms to some types of parallel architectures is equivalent to decomposing a transform $\mathbf{t}$ into a product of transforms $\mathbf{t} = \mathbf{t}^1 \boxed{\vee}\ \mathbf{t}^2 \boxed{\vee} \cdots \boxed{\vee}\ \mathbf{t}^k$, where each factor $\mathbf{t}^i$ is directly implementable on the parallel architecture. Since decomposing templates is the same as decomposing matrices, matrix decomposition techniques can be applied to template decomposition problems. Thus far, there exist no decomposition techniques for matrices under the matrix operation $\times$ as presented in section 1.2. Hence, the methods developed in Chapter 5 that decompose matrices are new results. They were developed mainly for solving the problem of mapping of transforms to particular parallel architectures, though they stand by themselves as a new theoretical result in the minimax algebra.

While some other areas of minimax algebra may seem to have no current applications to image processing, such as the eigenproblem, we present them in their image algebra form due to their interesting mathematical results.

# CHAPTER 3
# MAPPING OF MINIMAX ALGEBRA PROPERTIES TO IMAGE ALGEBRA PROPERTIES

This chapter is devoted to describing algebraic properties of the substructures $\{(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \mathbf{F}^{\mathbf{X}}, \boxed{\vee}, \vee, \boxed{\wedge}, \wedge\}$, where $\mathbf{F}$ is a subbelt of $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^{+}_{\pm\infty}$. During the investigation of the properties and before the discovery of the link to minimax algebra, many basic properties, such as the associativity of the $\boxed{\vee}$ operation, were proven within the context of the image algebra. Many theorems had excessive notational overhead, and often the proofs were laborious. Most of these same properties were found to have been stated and proven in context of the minimax algebra [38]. Using the matrix calculus makes some proofs less tedious, and in some cases makes them less cumbersome notationally. Thus, in order to place the presentation in a more elegant mathematical environment, we are omitting proofs that were done in the image algebra notation, and shall make use of the isomorphisms given in the previous chapter. Most of the theorems presented here are mapped into image algebra notation using the isomorphisms, and the proofs will be omitted. The results will be stated for both bounded l-groups, using the operations $\boxed{\vee}$ and $\textcircled{\vee}$.

## 3.1. Basic Definitions and Properties

Unless otherwise stated, we shall assume that $\mathbf{X}$, $\mathbf{Y}$, and $\mathbf{W}$ are finite coordinate sets, with $|\mathbf{X}| = m$, $|\mathbf{Y}| = n$, $|\mathbf{W}| = k$, with the pixel locations lexicographically ordered as in Chapter 2. The belt $\mathbf{F}$ with duality is a subbelt of either $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^{+}_{\pm\infty}$. The templates $\mathbf{s}$ and $\mathbf{t}$ will be $\mathbf{F}$ valued templates on appropriate domains, and $\mathbf{a}$, $\mathbf{b}$ will be $\mathbf{F}$ valued images. For the appropriate subbelt $\mathbf{F}$ of $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^{+}_{\pm\infty}$ according to the operation $\boxed{\vee}$ or $\textcircled{\vee}$, respectively, we have the following basic properties.

(1) $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \vee)$ is an s-lattice and $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \vee)$ is a function space over $(\mathbf{F}, \vee, \times)$;

(2) $\{(\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}, \vee, \boxed{\vee}\}$ is a belt; $\{(\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}, \vee, \ovee\}$ is a belt;

(3) $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \vee)$ is a left space over the belt $((\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}, \vee, \boxed{\vee})$; $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \vee)$ is a left space over the belt $((\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}, \vee, \ovee)$;

(4) $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ is a right space over the belt $\mathbf{F}$;

(5) We define *scalar multiplication* of a template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ by a scalar $\lambda \in \mathbf{F}$ as multiplication by the one-point template $\lambda \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ or $\lambda \in (\mathbf{F}_{-\infty}^{\mathbf{Y}})^{\mathbf{Y}}$, depending on whether the template $\lambda$ multiplies from the left or from the right, respectively, (and adjoining $-\infty$ to $\mathbf{F}$ if necessary), as

$$\mathbf{t} \boxed{\vee} \lambda = \lambda \boxed{\vee} \mathbf{t} = \mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}, \text{ where } \mathbf{s_y}(\mathbf{x}) = \mathbf{t_y}(\mathbf{x}) + \lambda$$

and

$$\mathbf{t} \ovee \lambda = \lambda \ovee \mathbf{t} = \mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}, \text{ where } \mathbf{s_y}(\mathbf{x}) = \mathbf{t_y}(\mathbf{x}) * \lambda.$$

Here, $\lambda_{\mathbf{y}}(\mathbf{x}) = \begin{cases} \lambda & \text{if } \mathbf{x} = \mathbf{y} \\ -\infty & \text{otherwise} \end{cases}$.

Next we state the distributive properties of $\boxed{\vee}$ and $\ovee$ with respect to $\vee$.

(6)

$$\begin{aligned}
&\mathbf{a} \boxed{\vee} (\mathbf{t} \vee \mathbf{s}) = (\mathbf{a} \boxed{\vee} \mathbf{t}) \vee (\mathbf{a} \boxed{\vee} \mathbf{s}) \\
&\mathbf{a} \boxed{\vee} (\mathbf{t} \boxed{\vee} \mathbf{s}) = (\mathbf{a} \boxed{\vee} \mathbf{t}) \boxed{\vee} \mathbf{s} \\
&(\mathbf{a} \vee \mathbf{b}) \boxed{\vee} \mathbf{t} = (\mathbf{a} \boxed{\vee} \mathbf{t}) \vee (\mathbf{b} \boxed{\vee} \mathbf{t}) \\
&(\mathbf{s} \vee \mathbf{t}) \boxed{\vee} \mathbf{u} = (\mathbf{s} \boxed{\vee} \mathbf{u}) \vee (\mathbf{t} \boxed{\vee} \mathbf{u}) \\
&\mathbf{u} \boxed{\vee} (\mathbf{s} \vee \mathbf{t}) = (\mathbf{u} \boxed{\vee} \mathbf{s}) \vee (\mathbf{u} \boxed{\vee} \mathbf{t}) \\
&\mathbf{s} \boxed{\vee} (\mathbf{t} \boxed{\vee} \mathbf{u}) = (\mathbf{s} \boxed{\vee} \mathbf{t}) \boxed{\vee} \mathbf{u}
\end{aligned}$$

$$\begin{aligned}
&\mathbf{a} \ovee (\mathbf{t} \vee \mathbf{s}) = (\mathbf{a} \ovee \mathbf{t}) \vee (\mathbf{a} \ovee \mathbf{s}) \\
&\mathbf{a} \ovee (\mathbf{t} \ovee \mathbf{s}) = (\mathbf{a} \ovee \mathbf{t}) \ovee \mathbf{s} \\
&(\mathbf{a} \vee \mathbf{b}) \ovee \mathbf{t} = (\mathbf{a} \ovee \mathbf{t}) \vee (\mathbf{b} \ovee \mathbf{t}) \\
&(\mathbf{s} \vee \mathbf{t}) \ovee \mathbf{u} = (\mathbf{s} \ovee \mathbf{u}) \vee (\mathbf{t} \ovee \mathbf{u}) \\
&\mathbf{u} \ovee (\mathbf{s} \vee \mathbf{t}) = (\mathbf{u} \ovee \mathbf{s}) \vee (\mathbf{u} \ovee \mathbf{t}) \\
&\mathbf{s} \ovee (\mathbf{t} \ovee \mathbf{u}) = (\mathbf{s} \ovee \mathbf{t}) \ovee \mathbf{u}.
\end{aligned}$$

The dual to properties 1 through 6 also hold, as both the belts $\mathbf{R}$ and $\mathbf{R}^+$ have duality.

(7) $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \wedge)$ is an s-lattice and $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}, \wedge)$ is a function space over $(\mathbf{F}, \wedge, \times')$;

(8) $\{(\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}, \wedge, \boxed{\wedge}\}$ is a belt. $\{(\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}, \wedge, \owedge\}$ is a belt.

etc.

Now let $\mathbf{F}$ be a subbelt of $\mathbf{R}$ or $\mathbf{R}^+$, and $\mathbf{F}_{\pm\infty}$ the bounded l-group with group $\mathbf{F}$. Corresponding to the identity matrix and the null matrix we have the *identity template* $\mathbf{l} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$, defined by

$$\mathbf{l}_{\mathbf{y}}(\mathbf{x}) = \begin{cases} \phi & \text{if } \mathbf{x} = \mathbf{y} \\ -\infty & \text{otherwise} \end{cases}$$

and the *null template* $\Phi \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ defined by

$$\Phi_{\mathbf{y}}(\mathbf{x}) = -\infty\,, \text{ for all } \mathbf{y} \in \mathbf{Y}, \mathbf{x} \in \mathbf{X}.$$

For the belt $\mathbf{R}$, $\phi = 0$, and for the belt $\mathbf{R}^+$, $\phi = 1$. Thus we have

$$\mathbf{a} \boxtimes \mathbf{l} = \mathbf{a}, \quad \mathbf{t} \boxtimes \mathbf{l} = \mathbf{l} \boxtimes \mathbf{t} = \mathbf{t} \;\; \forall\, \mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}, \forall\, \mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$$

For $\Phi \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}$,

$$\mathbf{t} \vee \Phi = \mathbf{t}, \quad \mathbf{t} \boxtimes \Phi = \Phi \boxtimes \mathbf{t} = \Phi, \quad \mathbf{a} \boxtimes \Phi = \text{null image}, \forall\, \mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}, \forall\, \mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}.$$

Similar properties hold for the operation $\circledast$.

### 3.1.1. Homomorphisms

We now discuss homomorphisms in context of the image algebra. Let $|\mathbf{X}| = \mathrm{m}$. Since the s-lattice $\{\mathbf{F}_{\pm\infty}^{\mathbf{X}}, \vee\}$ is isomorphic (via $\nu$) to the s-lattice $\{\mathbf{F}_{\pm\infty}^{\mathrm{m}}, \vee\}$, $\{\mathbf{F}_{\pm\infty}^{\mathbf{X}}, \vee\}$ is a space. For $\lambda \in \mathbf{F}^{\mathbf{X}}$ the constant image, we have

$$\mathbf{a} \vee \lambda = \lambda \vee \mathbf{a} = \mathbf{b} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}, \text{ where } \mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) \vee \lambda,$$

and for the one-point template $\lambda \in (\mathbf{R}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$,

$$\mathbf{a} \boxtimes \lambda = \lambda \boxtimes \mathbf{a} = \mathbf{b} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}, \text{ where } \mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \lambda,$$

if $\mathbf{F} = \mathbf{R}$, and

$$\mathbf{a} \circledast \lambda = \lambda \circledast \mathbf{a} = \mathbf{b} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}, \text{ where } \mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) * \lambda,$$

if $\mathbf{F} = \mathbf{R}^+$. Let $\mathbf{F} \in \{\mathbf{R}_{\pm\infty}, \mathbf{R}_{\pm\infty}^+\}$. Since $\{\mathbf{F}^{\mathbf{Y}}, \vee\}$ is an s-lattice, an s-lattice homomorphism

from $\mathbf{F}^{\mathbf{X}}$ to $\mathbf{F}^{\mathbf{Y}}$ is a function f: $\mathbf{F}^{\mathbf{X}} \rightarrow \mathbf{F}^{\mathbf{Y}}$ satisfying

$$f(\mathbf{a} \vee \mathbf{b}) = f(\mathbf{a}) \vee f(\mathbf{b}).$$

A right linear homomorphism g: $\mathbf{F}^{\mathbf{X}} \rightarrow \mathbf{F}^{\mathbf{Y}}$ is an s-lattice homomorphism satisfying

$$g(\mathbf{a} \boxed{\vee} \lambda) = g(\mathbf{a}) \boxed{\vee} \lambda.$$

Thus, the set of all right linear homomorphisms from $\mathbf{F}^{\mathbf{X}}$ to $\mathbf{F}^{\mathbf{Y}}$ is denoted by

$$\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^{\mathbf{X}}, \mathbf{F}^{\mathbf{Y}}) = \{g : \mathbf{F}^{\mathbf{X}} \rightarrow \mathbf{F}^{\mathbf{Y}}, \text{ and g satisfies } g(\mathbf{a} \vee \mathbf{b}) = g(\mathbf{a}) \vee g(\mathbf{b}), g(\mathbf{a} \boxed{\vee} \lambda) = g(\mathbf{a}) \boxed{\vee} \lambda\},$$

or if $\mathbf{F}$ is $\mathbf{R}^{+}$, then

$$\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^{\mathbf{X}}, \mathbf{F}^{\mathbf{Y}}) = \{g : \mathbf{F}^{\mathbf{X}} \rightarrow \mathbf{F}^{\mathbf{Y}}, \text{ and g satisfies } g(\mathbf{a} \vee \mathbf{b}) = g(\mathbf{a}) \vee g(\mathbf{b}), g(\mathbf{a} \ovee \lambda) = g(\mathbf{a}) \ovee \lambda\}.$$

### 3.1.2. Classification of Homomorphisms in the Image Algebra

Right linear transformations can be characterized entirely in terms of template transformations, and we give necessary and sufficient conditions for $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ to be isomorphic to $\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^{\mathbf{X}}, \mathbf{F}^{\mathbf{Y}})$.

**Theorem 3.1.** *Let* $\mathbf{F}$ *be a belt with identity and null element. Then for all non-empty finite coordinate sets* $\mathbf{X}$, $\mathbf{Y}$, $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ *is isomorphic to* $\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^{\mathbf{X}}, \mathbf{F}^{\mathbf{Y}})$.

**Corollary 3.2.** *Let* $\mathbf{F}$ *be a belt, and let* $\mathbf{X} \neq \emptyset$ *be a finite coordinate set with* $|\mathbf{X}| > 1$. *Then a necessary and sufficient condition that* $(\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ *be isomorphic to* $\mathrm{Hom}_{\mathbf{F}}(\mathbf{F}^{\mathbf{X}}, \mathbf{F}^{\mathbf{Y}})$, *for all non-empty finite coordinate sets* $\mathbf{Y}$, *is that* $\mathbf{F}$ *have an identity element* $\phi$ *with respect to* $\times$ *and a null element* $\theta$ *with respect to* $\vee$.

We call a template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ used with the operation $\vee$, $\boxed{\vee}$, $\boxed{\wedge}$, $\ovee$, or $\owedge$ a *lattice transform*. We will present an example of a transformation which is not right linear in section 6.1.

3.1.3. Inequalities

Some useful inequalities are stated in the next theorem.

**Theorem 3.3.** *Let* $\mathbf{F}$ *be a subbelt of* $\mathbf{R}_{\pm\infty}$ *or* $\mathbf{R}^{+}_{\pm\infty}$. *Then the following inequalities hold for images and templates with the appropriate domains, having values in* $\mathbf{F}$.

(i) $\mathbf{a} \vee (\mathbf{b} \wedge \mathbf{c}) \leq (\mathbf{a} \vee \mathbf{b}) \wedge (\mathbf{a} \vee \mathbf{c})$

(ii) $\mathbf{a} \wedge (\mathbf{b} \vee \mathbf{c}) \geq (\mathbf{a} \wedge \mathbf{b}) \vee (\mathbf{a} \wedge \mathbf{c})$

(iii) $(\mathbf{a} \wedge \mathbf{b}) \boxed{\vee} \mathbf{t} \leq (\mathbf{a} \boxed{\vee} \mathbf{t}) \wedge (\mathbf{b} \boxed{\vee} \mathbf{t})$ $\quad$ $(\mathbf{a} \wedge \mathbf{b}) \ovee \mathbf{t} \leq (\mathbf{a} \ovee \mathbf{t}) \wedge (\mathbf{b} \ovee \mathbf{t})$

(iv) $\mathbf{a} \boxed{\vee} (\mathbf{t} \wedge \mathbf{s}) \leq (\mathbf{a} \boxed{\vee} \mathbf{t}) \wedge (\mathbf{a} \boxed{\vee} \mathbf{s})$ $\quad$ $\mathbf{a} \ovee (\mathbf{t} \wedge \mathbf{s}) \leq (\mathbf{a} \ovee \mathbf{t}) \wedge (\mathbf{a} \ovee \mathbf{s})$

(v) $(\mathbf{a} \vee \mathbf{b}) \boxed{\wedge} \mathbf{t} \geq (\mathbf{a} \boxed{\wedge} \mathbf{t}) \vee (\mathbf{b} \boxed{\wedge} \mathbf{t})$ $\quad$ $(\mathbf{a} \vee \mathbf{b}) \owedge \mathbf{t} \geq (\mathbf{a} \owedge \mathbf{t}) \vee (\mathbf{b} \owedge \mathbf{t})$

(vi) $\mathbf{a} \boxed{\wedge} (\mathbf{t} \vee \mathbf{s}) \geq (\mathbf{a} \boxed{\wedge} \mathbf{t}) \vee (\mathbf{a} \boxed{\wedge} \mathbf{s})$ $\quad$ $\mathbf{a} \owedge (\mathbf{t} \vee \mathbf{s}) \geq (\mathbf{a} \owedge \mathbf{t}) \vee (\mathbf{a} \owedge \mathbf{s})$

(i) $\mathbf{s} \vee (\mathbf{t} \wedge \mathbf{r}) \leq (\mathbf{s} \vee \mathbf{t}) \wedge (\mathbf{s} \vee \mathbf{r})$

(ii) $\mathbf{s} \wedge (\mathbf{t} \vee \mathbf{r}) \geq (\mathbf{s} \wedge \mathbf{t}) \vee (\mathbf{s} \wedge \mathbf{r})$

(iii) $\mathbf{t} \boxed{\vee} (\mathbf{s} \wedge \mathbf{r}) \leq (\mathbf{t} \boxed{\vee} \mathbf{s}) \wedge (\mathbf{t} \boxed{\vee} \mathbf{r})$ $\quad$ $\mathbf{t} \ovee (\mathbf{s} \wedge \mathbf{r}) \leq (\mathbf{t} \ovee \mathbf{s}) \wedge (\mathbf{t} \ovee \mathbf{r})$

(iv) $(\mathbf{s} \wedge \mathbf{r}) \boxed{\vee} \mathbf{t} \leq (\mathbf{s} \boxed{\vee} \mathbf{t}) \wedge (\mathbf{r} \boxed{\vee} \mathbf{t})$ $\quad$ $(\mathbf{s} \wedge \mathbf{r}) \ovee \mathbf{t} \leq (\mathbf{s} \ovee \mathbf{t}) \wedge (\mathbf{r} \ovee \mathbf{t})$

(v) $\mathbf{t} \boxed{\wedge} (\mathbf{s} \vee \mathbf{r}) \geq (\mathbf{t} \boxed{\wedge} \mathbf{s}) \vee (\mathbf{t} \boxed{\wedge} \mathbf{r})$ $\quad$ $\mathbf{t} \owedge (\mathbf{s} \vee \mathbf{r}) \geq (\mathbf{t} \owedge \mathbf{s}) \vee (\mathbf{t} \owedge \mathbf{r})$

(vi) $(\mathbf{s} \vee \mathbf{r}) \boxed{\wedge} \mathbf{t} \geq (\mathbf{s} \boxed{\wedge} \mathbf{t}) \vee (\mathbf{r} \boxed{\wedge} \mathbf{t})$ $\quad$ $(\mathbf{s} \vee \mathbf{r}) \owedge \mathbf{t} \geq (\mathbf{s} \owedge \mathbf{t}) \vee (\mathbf{r} \owedge \mathbf{t})$

$$\mathbf{a} \boxed{\vee} (\mathbf{s} \boxed{\wedge} \mathbf{r}) \leq (\mathbf{a} \boxed{\vee} \mathbf{s}) \boxed{\wedge} \mathbf{r} \text{ and } \mathbf{a} \boxed{\wedge} (\mathbf{s} \boxed{\vee} \mathbf{r}) \geq (\mathbf{a} \boxed{\wedge} \mathbf{s}) \boxed{\vee} \mathbf{r}$$

$$\mathbf{t} \boxed{\vee} (\mathbf{s} \boxed{\wedge} \mathbf{r}) \leq (\mathbf{t} \boxed{\vee} \mathbf{s}) \boxed{\wedge} \mathbf{r} \text{ and } \mathbf{t} \boxed{\wedge} (\mathbf{s} \boxed{\vee} \mathbf{r}) \geq (\mathbf{t} \boxed{\wedge} \mathbf{s}) \boxed{\vee} \mathbf{r}$$

and

$$\mathbf{a} \ovee (\mathbf{s} \owedge \mathbf{r}) \leq (\mathbf{a} \ovee \mathbf{s}) \owedge \mathbf{r} \text{ and } \mathbf{a} \owedge (\mathbf{s} \ovee \mathbf{r}) \geq (\mathbf{a} \owedge \mathbf{s}) \ovee \mathbf{r}.$$

$$\mathbf{t} \ovee (\mathbf{s} \owedge \mathbf{r}) \leq (\mathbf{t} \ovee \mathbf{s}) \owedge \mathbf{r} \text{ and } \mathbf{t} \owedge (\mathbf{s} \ovee \mathbf{r}) \geq (\mathbf{t} \owedge \mathbf{s}) \ovee \mathbf{r}.$$

We remark that the above properties corresponding to the *forward* multiplications of an image by a template as defined in Chapter 1 are also valid, namely,

$\mathbf{t} \boxtimes (\mathbf{a} \wedge \mathbf{b}) \leq (\mathbf{t} \boxtimes \mathbf{a}) \wedge (\mathbf{t} \boxtimes \mathbf{b})$, etc.

3.1.4. Conjugacy

The notion of conjugacy as discussed in section 1.2 extends to templates as well. Suppose that $\mathbf{F}$ and $\mathbf{F}^*$ are conjugate. Then for $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, $\mathbf{t}^* \in ((\mathbf{F}^*)^{\mathbf{Y}})^{\mathbf{X}}$ is defined by

$$\mathbf{t}^*_{\mathbf{x}}(\mathbf{y}) \equiv (\mathbf{t}_{\mathbf{y}}(\mathbf{x}))^*.$$

The conjugate of $\mathbf{t} \in (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$ is the additive dual $\mathbf{t}^*$, and the conjugate of $\mathbf{t} \in ((\mathbf{R}^{+}_{\pm\infty})^{\mathbf{X}})^{\mathbf{Y}}$ is the multiplicative dual $\bar{\mathbf{t}}$, both of which are defined in section 1.1.

Let P be any set of $\mathbf{F}$ valued templates from $\mathbf{Y}$ to $\mathbf{X}$, with $\mathbf{F}$ and $\mathbf{F}^*$ as conjugate systems. Define $\mathrm{P}^*$ by

$$\mathrm{P}^* \equiv \{\mathbf{t}^* : \mathbf{t} \in \mathrm{P}\}.$$

Here, the star symbol * denotes the dual template for either value set $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^{+}_{\pm\infty}$. Note that $\mathrm{P}^* \subset ((\mathbf{F}^*)^{\mathbf{Y}})^{\mathbf{X}}$. We have

**Theorem 3.4**. *Let* $(\mathbf{F},\vee)$ *and* $(\mathbf{F}^*,\wedge)$ *be conjugate. Then* $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}},\vee,\boxtimes)$ *and* $(((\mathbf{F}^*)^{\mathbf{Y}})^{\mathbf{X}},\wedge,\boxtimes)$ *are conjugate, where* $\mathbf{F}$ *is a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$, *and* $((\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}},\vee,\circledvee)$ *and* $(((\mathbf{F}^*)^{\mathbf{Y}})^{\mathbf{X}},\wedge,\circledwedge)$ *are conjugate, where* $\mathbf{F}$ *is a sub-bounded l-group of* $\mathbf{R}^{+}_{\pm\infty}$, *for any non-empty finite coordinate sets* $\mathbf{X}$, $\mathbf{Y}$. *In all cases the conjugate of a given template* $\mathbf{t}$ *is the dual template* $\mathbf{t}^*$ *or* $\bar{\mathbf{t}}$ *of the respective bounded l-group as defined in Chapter* 1.

**Proposition 3.5**. *If* $(\mathbf{F},\vee,\times,\wedge,\times')$ *is a self-conjugate belt, then* $((\mathbf{F}^*)^{\mathbf{X}})^{\mathbf{Y}} = (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ *for all non-empty finite coordinate sets* $\mathbf{X}$, $\mathbf{Y}$. *Also,* $((\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}},\vee,\boxtimes,\wedge,\boxtimes)$ *is a self-conjugate belt, and* $(((\mathbf{R}^{+}_{\pm\infty})^{\mathbf{X}})^{\mathbf{X}},\vee,\circledvee,\wedge,\circledwedge)$ *is a self-conjugate belt.*

An example. In this section we give an application to a scheduling problem, showing the use of the conjugate of a template. In particular, this example provides a physical interpretation of the conjugate of a template.

Suppose we have n tasks, or activities, or subroutines, labelled 1,...,n. Let $\mathbf{a}(\mathbf{x}_i)$ denote the starting time of task i, and assume without loss of generality that task 1 is the starting activity, task n is the finishing activity, and that tasks 2 through n-1 are intermediate activities. Suppose we are given the time of the starting activity, and we wish to know the soonest time at which each subsequent activity can be started. In particular, what is the earliest time that task n can start, or, what is the earliest expected time of completion of the collection of tasks?

The relation of the tasks to one another can be described by a partial order $\mathcal{R}$ on the set of tasks {1,...,n}:

j $\mathcal{R}$ i if and only if task j is to be completed before task i can start.

Let $d_{ij}$ denote the minimum amount of time by which the start of activity j must precede the start of activity i. That is, $d_{ij}$ is the the duration time of activity j, or the processing time of task j, which must pass before activity i can start. Define $\mathbf{w} \in (\mathbf{R}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ by

$$\mathbf{w}_{\mathbf{x}_i}(\mathbf{x}_j) = \begin{cases} d_{ij} & \text{if } j \,\mathcal{R}\, i \\ -\infty & \text{otherwise} \end{cases}.$$

There is an obvious relationship between the weighted digraph associated with the partial order relation $\mathcal{R}$ and the template $\mathbf{w}$. For example, suppose we have 5 tasks or activities, or subroutines of a program, which have the following relation or partial order:

(1,2) (1,3) (2,4) (2,5) (3,4) (3,5) (4,5)

Here, activity 1 is the start activity, activity 5 is the end activity, and tasks 2,3,4 are inter-

mediate tasks or subroutines. Suppose the duration times $d_{ij}$ of the activities are:

$$d_{21} = 1 \quad d_{31} = 6 \quad d_{42} = 2$$

$$d_{43} = 1 \quad d_{52} = 1 \quad d_{53} = 3 \quad d_{54} = 3$$

and $d_{ii} = 0$ for each $i = 1,...,5$. This is consistent with a meaningful physical interpretation of the definition of duration time for a task.

The corresponding weighted digraph is given in Figure 6.

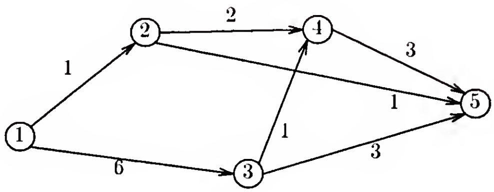



**Figure 6.** A Scheduling Network.

The nodes represent the activities, and the duration times are given as numbers on the directed edges linking the nodes.

In determining $\mathbf{a}(\mathbf{x}_4)$ for example, note that $\mathbf{a}(\mathbf{x}_4)$ must satisfy

$$\mathbf{a}(\mathbf{x}_4) \;=\; \max\{d_{42} + \mathbf{a}(\mathbf{x}_2),\, d_{43} + \mathbf{a}(\mathbf{x}_3),\, d_{44} + \mathbf{a}(\mathbf{x}_4)\}$$

or equivalently

$$\mathbf{a}(\mathbf{x}_4) \;=\; \max_{1 \le j \le 5}\{\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_j) + \mathbf{a}(\mathbf{x}_j)\}.$$

This last equality follows from the fact that $\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_j) = -\infty$ if j is not related to i. In the general setting, we must solve, for each $i = 1,...,n$:

$$\mathbf{a}(\mathbf{x}_i) = \max_{1 \le j \le n} \{ \mathbf{w}_{\mathbf{x}_i}(\mathbf{x}_j) + \mathbf{a}(\mathbf{x}_j) \}$$

or, writing the problem as an image algebra expression, we must solve for **a** in

$$\mathbf{a} \boxed{\vee} \mathbf{w} = \mathbf{a}. \tag{3-1}$$

Here, **a** is an image on $\mathbf{X}$ where $|\mathbf{X}| = n$.

An analysis of a network in this manner is called *backward recursion analysis.*

Under *forward recursion*, suppose we have n tasks with duration times $f_{ij}$, where $f_{ij}$ is the minimum amount of time by which the start of activity i must precede the start of activity j, if the activities are so related. Otherwise, let $f_{ij}$ have value $-\infty$. Define $\mathbf{w} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ by

$$\mathbf{w}_{\mathbf{x}_i}(\mathbf{x}_j) = \begin{cases} f_{ij} & \text{if } j \mathcal{R} i \\ -\infty & \text{otherwise} \end{cases}.$$

As before, $f_{ii} = 0$ gives a consistent physical interpretation.

Let $\tau$ be the planned completion date of the project, which is given, and define $\mathbf{a}(\mathbf{x}_i)$ to be the latest allowable starting time for activity i. We wish to determine $\mathbf{a}(\mathbf{x}_1),\ldots,\mathbf{a}(\mathbf{x}_{n-1})$ such that $\mathbf{a}(\mathbf{x}_n) = \tau$. Thus, we desire to solve for **a** in

$$\mathbf{a}(\mathbf{x}_i) = \min_{j=1,\ldots,n} (-\mathbf{w}_{\mathbf{x}_i}(\mathbf{x}_j) + \mathbf{a}(\mathbf{x}_j))$$

for $i = 1,\ldots,n$. For example, for 5 nodes, suppose we have the following relations:

$$(1,2) \ (1,3) \ (2,4) \ (2,5) \ (3,4) \ (3,5) \ (4,5) .$$

Here we write (i,j) if task i must precede task j. Suppose the times $f_{ij}$ of the activities are:

$$f_{12} = 1 \quad f_{13} = 6 \quad f_{24} = 2$$

$$f_{34} = 1 \quad f_{25} = 1 \quad f_{35} = 3 \quad f_{45} = 3$$

Suppose we would like to find $\mathbf{a}(\mathbf{x}_4)$, say, satisfying

$$\mathbf{a}(\mathbf{x}_4) \;=\; \min_{j=1,\ldots,5}\left(-\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_j) + \mathbf{a}(\mathbf{x}_j)\right).$$

The value $-\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_5) + \mathbf{a}(\mathbf{x}_5)$ is the latest allowable time to start task 5 minus the minimum amount of time activity 4 must precede activity 5, and the time to start task 4 must be at least as small as this number. Thus, the time to start task 5 must be at least as small as -1 $+ \mathbf{a}(\mathbf{x}_5)$. The value $\mathbf{a}(\mathbf{x}_4) = \min\{-\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_5) + \mathbf{a}(\mathbf{x}_5)\} = -1 + \tau$. (All other values $-\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_j) + \mathbf{a}(\mathbf{x}_j) = +\infty$ as $-\mathbf{w}_{\mathbf{x}_4}(\mathbf{x}_j) = +\infty$ for $j \neq 5$.) Since $\tau$ is given, this quantity can be explicitly determined. The remaining equations can be solved similarly.

If we define $\mathbf{u} \in (\mathbf{R}_{+\infty}^{\mathbf{X}})^{\mathbf{X}}$ by

$$\mathbf{u}_{\mathbf{x}_i}(\mathbf{x}_j) \;=\; \begin{cases} -\mathbf{w}_{\mathbf{x}_j}(\mathbf{x}_j) & \text{if } j \mathcal{R} i \\ +\infty & \text{otherwise} \end{cases}$$

then it is obvious that in general we must solve for $\mathbf{a}$ the following:

$$\mathbf{a} \boxtimes \mathbf{u} \;=\; \mathbf{a}. \tag{3-2}$$

It is clear that the template $\mathbf{u}$ in equation (3-2) is the conjugate of the template $\mathbf{w}$ in Equation (3-1). That is,

$$\mathbf{u} \;=\; \mathbf{w}^*.$$

We can say that the templates $\mathbf{w}$ and $\mathbf{w}^*$ define the structure of the network as we analyze it backward or forward in time, respectively.

### 3.1.5. Alternating $\mathbf{tt}^*$ and $\mathbf{t\bar{t}}$ Products

This section discusses the concept of an *alternating* $\mathbf{tt}^*$ or $\mathbf{t\bar{t}}$ *product* of a template $\mathbf{t}$ and its conjugate under the operation $\boxtimes$ or $\circledvee$, respectively. We shall state the results only for the sub-bounded l-groups of $\mathbf{R}_{\pm\infty}$ and the operations $\boxtimes$ and $\boxtimes$, with the understanding that unless otherwise stated, an arbitrary sub-bounded l-group of $\mathbf{R}_{\pm\infty}^{+}$ and the operations $\circledvee$ and $\circledwedge$ may be substituted in the appropriate places.

**Theorem 3.6.** *Let* $\mathbf{F}_{\pm\infty}$ *be a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$, *where* $\mathbf{F}$ *denotes the group of the bounded l-group* $\mathbf{F}_{\pm\infty}$, *and* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{Y}}$. *Then we have*

$$\mathbf{t} \boxed{\vee} (\mathbf{t}^* \boxed{\wedge} \mathbf{t}) = \mathbf{t} \boxed{\wedge} (\mathbf{t}^* \boxed{\vee} \mathbf{t}) = (\mathbf{t} \boxed{\wedge} \mathbf{t}^*) \boxed{\vee} \mathbf{t} = (\mathbf{t} \boxed{\vee} \mathbf{t}^*) \boxed{\wedge} \mathbf{t} = \mathbf{t}.$$

*Similarly,*

$$\mathbf{t}^* \boxed{\vee} (\mathbf{t} \boxed{\wedge} \mathbf{t}^*) = \mathbf{t}^* \boxed{\wedge} (\mathbf{t} \boxed{\vee} \mathbf{t}^*) = (\mathbf{t}^* \boxed{\wedge} \mathbf{t}) \boxed{\vee} \mathbf{t}^* = (\mathbf{t}^* \boxed{\vee} \mathbf{t}) \boxed{\wedge} \mathbf{t}^* = \mathbf{t}^*.$$

We now define an *alternating* $\mathbf{tt}^*$ *product.* Write a *word* consisting of the letters $\mathbf{t}$ and $\mathbf{t}^*$, in an alternating sequence. A single letter $\mathbf{t}$ or $\mathbf{t}^*$ is allowed. If we have $k > 1$ letters, now insert k-1 symbols of $\boxed{\vee}$ and $\boxed{\wedge}$, in an alternating manner. For example, the following sequences are allowed:

$$\mathbf{t}^* \boxed{\wedge} \mathbf{t}$$

$$\mathbf{t} \boxed{\wedge} \mathbf{t}^* \boxed{\vee} \mathbf{t}$$

$$\mathbf{t}^* \boxed{\vee} \mathbf{t} \boxed{\wedge} \mathbf{t}^* \boxed{\vee} \mathbf{t} \boxed{\wedge} \mathbf{t}^* \boxed{\vee} \mathbf{t}.$$

Now insert brackets in an arbitrary way so that the resulting expression is not ambiguous. For example,

$$\mathbf{t}^* \boxed{\wedge} \mathbf{t}$$

$$\mathbf{t} \boxed{\wedge} (\mathbf{t}^* \boxed{\vee} \mathbf{t})$$

$$(\mathbf{t}^* \boxed{\vee} ((\mathbf{t} \boxed{\wedge} \mathbf{t}^*) \boxed{\vee} \mathbf{t})) \boxed{\wedge} (\mathbf{t}^* \boxed{\vee} \mathbf{t}).$$

Any algebraic expression so constructed is called an *alternating* $\mathbf{tt}^*$ *product.*

Suppose an alternating $\mathbf{tt}^*$ product an odd number of letters $\mathbf{t}$ and/or $\mathbf{t}^*$. Then we say it is of *type* $\mathbf{t}$ if it begins and ends with an $\mathbf{t}$, and that it is of *type* $\mathbf{t}^*$ if it begins and ends with an $\mathbf{t}^*$. If it has an even number of letters we say that it is of *type*

$$\mathbf{t} \boxed{\vee} \mathbf{t}^* \text{ or } \mathbf{t} \boxed{\wedge} \mathbf{t}^* \text{ or } \mathbf{t}^* \boxed{\wedge} \mathbf{t}$$

exactly according to the first two letters with its separating operator, regardless of how the

brackets lie in the entire expression. As an example:

$$\begin{array}{lll} \mathbf{t}^* \boxed{\wedge}\, \mathbf{t} & \text{is of type} & \mathbf{t}^* \boxed{\wedge}\, \mathbf{t} \\ \mathbf{t} \boxed{\wedge} (\mathbf{t}^* \boxed{\vee}\, \mathbf{t}) & \text{is of type} & \mathbf{t} \\ (\mathbf{t}^* \boxed{\vee} ((\mathbf{t} \boxed{\wedge}\, \mathbf{t}^*) \boxed{\vee}\, \mathbf{t})) \boxed{\wedge} (\mathbf{t}^* \boxed{\vee}\, \mathbf{t}) & \text{is of type} & \mathbf{t}^* \boxed{\vee}\, \mathbf{t}. \end{array}$$

**Theorem 3.7.** *Let* $\mathbf{F}_{\pm\infty}$ *be a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$, *and* $\mathbf{t}$ *an arbitrary template in* $(\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$. *Then every alternating* $\mathbf{t}\mathbf{t}^*$ *product* P *is well-defined, and if* P *is of type* Q, *then* P $=$ Q.

If a product P has more than 1 letter, then we define P($\mathbf{z}$) to be the formal product obtained when the last (rightmost) letter, $\mathbf{t}$ or $\mathbf{t}^*$ (or $\bar{\mathbf{t}}$), is replaced by $\mathbf{z}$, where $\mathbf{z}$ is a $\mathbf{F}$ valued template on the appropriate coordinate sets $\mathbf{X}$ and $\mathbf{Y}$.

**Theorem 3.8.** *Let* $\mathbf{F}_{\pm\infty}$ *be a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$, *and* $\mathbf{t}$, $\mathbf{z}$ *arbitrary templates over* $\mathbf{F}$. *If* P *is an alternating* $\mathbf{t}\mathbf{t}^*$ *product containing four letters and* P *is of type* Q, *then*

$$\mathrm{P}(\mathbf{z}) = \mathrm{Q}(\mathbf{z}).$$

## 3.2. Systems of Equations

We now discuss the problem of finding solutions to the problem:

Given $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ and $\mathbf{b} \in \mathbf{R}_{\pm\infty}^{\mathbf{Y}}$, find $\mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}$ such that $\mathbf{a} \boxed{\vee}\, \mathbf{t} = \mathbf{b}$. (3-3)

Similarly, we also wish to solve:

Given $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{Y}}$ and $\mathbf{b} \in (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{Y}}$, find $\mathbf{a} \in (\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}}$ such that $\mathbf{a} \circledvee \mathbf{t} = \mathbf{b}$.

Here, $|\mathbf{X}| = \mathrm{m}$, $|\mathbf{Y}| = \mathrm{n}$.

### 3.2.1 F-asticity and /-solutions

If $\mathbf{F}$ is a bounded l-group and x, y $\in \mathbf{F}$, we say that the products x $\times$ y and x $\times'$ y are */-undefined* if one of x,y is $-\infty$ and the other is $+\infty$. We say that a template product is */-undefined* if the evaluation of $\mathbf{t}_{\mathbf{y}}(\mathbf{x})$ requires the formation of a /-undefined product of elements of the bounded l-group $\mathbf{F}_{\pm\infty}$. Otherwise, we say that a template product is */-defined* or */-exists.* Some mathematical models require solutions which avoid the formation of /-undefined products, as in practical cases these often correspond to unrelated activities. We state these results for both bounded l-groups where appropriate, with the $\circledvee$ results in parentheses. As usual, the sub-bounded l-group $\mathbf{F}_{\pm\infty}$ is dependent on which operation, $\boxed{\vee}$ or $\circledvee$ is used.

**Lemma 3.9.** *Let* $\mathbf{F}_{\pm\infty}$ *be a subbelt of* $\mathbf{R}_{\pm\infty}$ $(\mathbf{R}^{+}_{\pm\infty})$. *Let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be non-empty, finite arrays, and* $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$. *Then the set of all images* $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}_{\pm\infty}$ *such that* $\mathbf{a} \boxed{\vee} \mathbf{t}$ $(\mathbf{a} \circledvee \mathbf{t})$ *is /-defined is a sub-s-lattice of* $\mathbf{F}^{\mathbf{X}}_{\pm\infty}$. *Hence the set of solutions* $\mathbf{a}$ *of statement* (3-3) *such that* $\mathbf{a} \boxed{\vee} \mathbf{t}$ $(\mathbf{a} \circledvee \mathbf{t})$ */-exists is either empty or is a sub-s-lattice of* $\mathbf{F}^{\mathbf{X}}_{\pm\infty}$.

**Lemma 3.10.** *Let* $\mathbf{X}$, $\mathbf{Y}$, *and* $\mathbf{W}$ *be non-empty, finite arrays, and* $\mathbf{t} \in (\mathbf{F}^{\mathbf{W}}_{\pm\infty})^{\mathbf{Y}}$. *Then the set of templates* $\mathbf{s} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{W}}$, *such that* $\mathbf{s} \boxed{\vee} \mathbf{t}$ $(\mathbf{s} \circledvee \mathbf{t})$ *is /-defined is a sub-s-lattice of* $(\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$.

Any solution $\mathbf{a}$ of statement (3-3) such that $\mathbf{a} \boxed{\vee} \mathbf{t}$ $(\mathbf{a} \circledvee \mathbf{t})$ /-exists is called a */-solution* of (3-3).

**Theorem 3.11.** *Let* $\mathbf{F}_{\pm\infty}$ *be a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$ $(\mathbf{R}^{+}_{\pm\infty})$. *Then* (3-3) *has at least one solution if and only if* $\mathbf{a} = \mathbf{b} \boxed{\wedge} \mathbf{t}^{*}$ $(\mathbf{a} = \mathbf{b} \circledwedge \bar{\mathbf{t}})$ *is a solution. In this case,* $\mathbf{a} = \mathbf{b} \boxed{\wedge} \mathbf{t}^{*}$ $(\mathbf{a} = \mathbf{b} \circledwedge \bar{\mathbf{t}})$ *is the greatest solution.*

Recall from probability theory that a row-stochastic matrix is a non-negative matrix in which the sum of the elements in each row is equal to 1. We will make analogous definitions, where the operation $+$ is replaced by the operation $\vee$, and the unity element is $-\infty$.

Let $P \subset \mathbf{F}_{\pm\infty}$, where $\mathbf{F}_{\pm\infty}$ is an arbitrary sub-bounded l-group of $\mathbf{R}_{\pm\infty}$ ($\mathbf{R}^{+}_{\pm\infty}$). A template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$ is called *row*-P-*astic* if $\bigvee_{j=1}^{n} \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j) \in P$ for all $i = 1,\ldots,n$ and *column*-P-*astic* if $\bigvee_{i=1}^{n} \mathbf{t}'_{\mathbf{x}_j}(\mathbf{y}_i) \in P$ for all $j = 1,\ldots,m$. The template $\mathbf{t}$ is called *doubly*-P-*astic* if $\mathbf{t}$ is both row- and column-P-astic. Note that if $\mathbf{t}$ is column-P-astic, then $\mathbf{t}'$ is row-P-astic.

**Theorem 3.12.** *Let* $\mathbf{F}_{\pm\infty}$ *be a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$ *(*$\mathbf{R}^{+}_{\pm\infty}$*) and* $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$, $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}_{\pm\infty}$ *such that* (3-3) *is soluble. Then* $\mathbf{a} = \mathbf{b} \boxed{\wedge} \mathbf{t}^{*}$ $(\mathbf{a} = \mathbf{b} \circledwedge \bar{\mathbf{t}})$ */-exists and is a /-solution of* (3-3)*, if and only if one of the following cases is satisfied:*

(i) $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, and $\mathbf{b} = +\infty$, the constant image with $+\infty$ everywhere.
(ii) $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, and $\mathbf{b} = -\infty$.
(iii) $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$ is doubly $\mathbf{F}$-astic, and $\mathbf{b} \in \mathbf{F}^{\mathbf{X}}$.

*Moreover, every solution of* (3-3) *is then a /-solution, and* $\mathbf{b} \boxed{\wedge} \mathbf{t}^{*}$ $(\mathbf{b} \circledwedge \bar{\mathbf{t}})$ *is equal to* $+\infty$, $-\infty$, *or is finite, respectively according as case* (i), (ii), *or* (iii) *holds.*

In the following theorem, we state the dual and left-right generalizations of Theorems 3.11 and 3.12.

**Corollary 3.13.** *Let* $\mathbf{F}_{\pm\infty}$ *be a sub-bounded l-group of* $\mathbf{R}_{\pm\infty}$ *(*$\mathbf{R}^{+}_{\pm\infty}$*), and let* $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$, $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}_{\pm\infty}$. *Then for all combinations of* $\mathbf{c}$,$\mathbf{q}$, *and* $\delta$ *given in Table* 1, *the following statement is true:*

> *The image algebra equation* $\mathbf{c}$ *has at least one solution if and only if the product* $\mathbf{d}$ *is a solution; and the product* $\mathbf{d}$ *is then the* $\delta$ *solution. Furthermore, if the product* $\mathbf{d}$ *is /-defined, and equation* $\mathbf{c}$ *is /-defined when* $\mathbf{a} = \mathbf{d}$, *then equation* $\mathbf{c}$ *is /-defined when* $\mathbf{a}$ *is any solution of equation* $\mathbf{c}$. *If* $\mathbf{F}_{\pm\infty}$ *is a sub-bounded l-group of* $\mathbf{R}^{+}_{\pm\infty}$, *then*

*the results in Table 1 hold for* $\ovee$ *replacing* $\boxed{\vee}$ *everywhere and* $\bar{\mathbf{t}}$ *replacing* $\mathbf{t}^*$ *everywhere.*

**Table 1.**

| $\mathbf{c}$ | $\mathbf{d}$ | $\delta$ |
|---|---|---|
| $\mathbf{a}\ \boxed{\vee}\ \mathbf{t}=\mathbf{b}$ | $\mathbf{b}\ \boxed{\wedge}\ \mathbf{t}^*$ | greatest |
| $\mathbf{a}\ \boxed{\vee}\ \mathbf{t}^*=\mathbf{b}$ | $\mathbf{b}\ \boxed{\wedge}\ \mathbf{t}$ | greatest |
| $\mathbf{a}\ \boxed{\wedge}\ \mathbf{t}=\mathbf{b}$ | $\mathbf{b}\ \boxed{\vee}\ \mathbf{t}^*$ | least |
| $\mathbf{a}\ \boxed{\wedge}\ \mathbf{t}^*=\mathbf{b}$ | $\mathbf{b}\ \boxed{\vee}\ \mathbf{t}$ | least |
| $\mathbf{t}\ \boxed{\vee}\ \mathbf{a}=\mathbf{b}$ | $\mathbf{t}^*\ \boxed{\wedge}\ \mathbf{b}$ | greatest |
| $\mathbf{t}^*\ \boxed{\vee}\ \mathbf{a}=\mathbf{b}$ | $\mathbf{t}\ \boxed{\wedge}\ \mathbf{b}$ | greatest |
| $\mathbf{t}\ \boxed{\wedge}\ \mathbf{a}=\mathbf{b}$ | $\mathbf{t}^*\ \boxed{\vee}\ \mathbf{b}$ | least |
| $\mathbf{t}^*\ \boxed{\wedge}\ \mathbf{a}=\mathbf{b}$ | $\mathbf{t}\ \boxed{\vee}\ \mathbf{b}$ | least |

If $\mathbf{d}$ is a solution to $\mathbf{c}$ in Table 1, then $\mathbf{d}$ is called a *principal solution.*

We can also restate the last three theorems as a solubility criterion.

**Problem (3-3)** is soluble if and only if $(\mathbf{b}\ \boxed{\wedge}\ \mathbf{t}^*)\ \boxed{\vee}\ \mathbf{t}=\mathbf{b}$ $[(\mathbf{b}\ \owedge\ \bar{\mathbf{t}})\ \ovee\ \mathbf{t})=\mathbf{b}]$; and every solution is a /-solution if $(\mathbf{b}\ \boxed{\wedge}\ \mathbf{t}^*)\boxed{\vee}\ \mathbf{t}$ $[(\mathbf{b}\ \owedge\ \bar{\mathbf{t}})\ \ovee\ \mathbf{t})=\mathbf{b}]$ /-exists.

Note that Theorem 3.12 identifies the cases in which (3-3) has a /-defined /-solution. All solutions are then /-solutions. The next question to ask is: can we find all solutions? We now focus on the following problem.

Given that $\mathbf{F}$ is $\mathbf{R}_{\pm\infty}(\mathbf{R}^{+}_{\pm\infty})$ and that $(\mathbf{b} \boxtimes \mathbf{t}^{*}) \boxtimes \mathbf{t})$ $[(\mathbf{b} \circledwedge \bar{\mathbf{t}}) \circledvee \mathbf{t})]$ /-exists and equals $\mathbf{b}$, find all solutions of (3-3). (3-4)

For cases (i) and (ii) of Theorem 3.12, we note that $\mathbf{t}$ is finite. The next proposition gives solutions for these two cases.

**Proposition 3.14.** *Let $\mathbf{F}_{\pm\infty}$ be a sub-bounded l-group of $\mathbf{R}_{\pm\infty}(\mathbf{R}^{+}_{\pm\infty})$. If $\mathbf{b} = -\infty$ (the constant image), then Problem* (3-4) *has $\mathbf{b}$ as its unique solution. If $\mathbf{b} = +\infty$, then Problem* (3-4) *has as its solutions exactly those images of $\mathbf{F}^{\mathbf{X}}_{\pm\infty}$ which have at least one pixel value equal to $+\infty$.*

To determine solutions to case (iii), we need to consider the particular case that $\mathbf{F}_{\pm\infty}$ is the 3-element bounded l-group $\mathbf{F}_3$. Here $\mathbf{b}$ is finite with all elements having value $\phi$.

**Lemma 3.15.** *Let $\mathbf{F}_{\pm\infty}$ be the 3-element bounded l-group $\mathbf{F}_3$. Let $\mathbf{t}$ be doubly $\mathbf{F}$-astic and $\mathbf{b}$ be finite. Then* (3-3) *is soluble, having as principal /-solution $\mathbf{a} = \mathbf{l}$ where $\mathbf{l}(\mathbf{x}) = \phi$ for all i. Hence, no solution to* (3-3) *contains $+\infty$ for any pixel value, and all solutions are /-solutions.*

3.2.2. All Solutions to $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ and $\mathbf{a} \circledvee \mathbf{t} = \mathbf{b}$

We now give some criterions for finding all solutions to problem (3-3) for the case where the template $\mathbf{t}$ is doubly $\mathbf{F}$-astic and $\mathbf{b}$ finite. We discuss the general case where $\mathbf{F}$ is the belt $\mathbf{R}$ or $\mathbf{R}^{+}$.

If a template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}$ has form

$$\mathbf{t}_{\mathbf{x}_i}(\mathbf{x}_i) = \alpha_i, \text{ and } \mathbf{t}_{\mathbf{x}_i}(\mathbf{x}_j) = -\infty, \; j \neq i,$$

we write $\mathbf{t} = \operatorname{diag}(\alpha_1, \alpha_2, \ldots, \alpha_m)$.

For $\mathbf{b} \in \mathbf{F}$ finite, define the template $\mathbf{d} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ by $\mathbf{d} =$

$$\mathrm{diag}([\mathbf{b}(\mathbf{x}_1)]^*, [\mathbf{b}(\mathbf{x}_2)]^*, \ldots, [\mathbf{b}(\mathbf{x}_m)]^*).$$

Since $\mathbf{b}$ is finite, so is $\mathbf{d}_{\mathbf{x}_i}(\mathbf{x}_i)$, and $\mathbf{d}_{\mathbf{x}_i}(\mathbf{x}_i) = -\mathbf{b}(\mathbf{x}_i)$ (or $1/\mathbf{b}(\mathbf{x}_i)$) $\forall\ i = 1,...,m$. Thus, solving (3-3) is equivalent to solving

$$\mathbf{a} \boxtimes \mathbf{s} = \mathbf{1}, \tag{3-5}$$

or

$$\mathbf{a} \circledvee \mathbf{s} = \mathbf{1},$$

where $\mathbf{s} = \mathbf{d} \boxtimes \mathbf{t}$ ($\mathbf{s} = \mathbf{d} \circledvee \mathbf{t}$) $\in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$, and $\mathbf{1} = \phi$, the constant image. Note that $\mathbf{s}_{\mathbf{y}_k}(\mathbf{x}_j) = \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_j) - \mathbf{b}(\mathbf{x}_j)$ ($\mathbf{s}_{\mathbf{y}_k}(\mathbf{x}_j) = \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_j) * 1/\mathbf{b}(\mathbf{x}_j)$ ). Now, for each image $\mathbf{s}'_{\mathbf{x}_j} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$, let $W'_j = \{(\mathbf{x}_j, \mathbf{y}_i) : \mathbf{s}'_{\mathbf{x}_j}(\mathbf{y}_i) = \bigvee_{k=1}^{m} \mathbf{s}'_{\mathbf{x}_j}(\mathbf{y}_k)\ \}$. Note that $W'_j \subset \mathbf{X} \times \mathbf{Y}$ for every j. The elements $\mathbf{s}'_{\mathbf{x}_j}(\mathbf{y}_i)$ corresponding to $(\mathbf{x}_j, \mathbf{y}_i) \in W'_j$ are called *marked values* of $W'_j$. Notice that every image $\mathbf{s}'_{\mathbf{x}_j}$ will have at least one marked value, as $\mathbf{d}$, $\mathbf{t}$ and $\mathbf{s}$ are doubly $\mathbf{F}$-astic. Our next theorem gives conditions where there is no solution.

**Lemma 3.16.** *Let $\mathbf{F}_{\pm\infty}$ be a bounded l-group, $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}$ where $\mathbf{t}$ is doubly $\mathbf{F}$-astic, and $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}$. Define $\mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}$ by*

$$\mathbf{s} = \mathbf{d} \boxtimes \mathbf{t} \quad (\text{or} \quad \mathbf{s} = \mathbf{d} \circledvee \mathbf{t}\ )$$

*depending on whether the group $\mathbf{F}$ is $\mathbf{R}$ or $\mathbf{R}^+$, respectively, and $\mathbf{d}$ is as above. Suppose there exists i such that for no j is $\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j)$ a marked value. That is, suppose there exists $\mathbf{y}_i \in \mathbf{Y}$ such that $\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j)$ is not a marked value for any j. Then there does not exist $\mathbf{a} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$ such that $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ ($\mathbf{a} \circledvee \mathbf{t} = \mathbf{b}$).*

There now remains the case in which for every i, there is at least one j such that $\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j)$ is a marked value. We transform the question into a boolean problem, where it can be

shown that the following procedure will give a set of solutions to equation (3-5) [38].

**Step 1.** For the bounded l-group $\mathbf{F}_{\pm\infty} = \mathbf{F}_3$, define $\mathbf{g} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ by

$$\mathbf{g}_{\mathbf{y}_i}(\mathbf{x}_j) = \begin{cases} \phi & \text{if } \mathbf{s}'_{\mathbf{x}_j}(\mathbf{y}_i) \text{ is marked} \\ -\infty & \text{otherwise} \end{cases}.$$

Letting $\mathbf{f} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$, now solve the boolean system

$$\mathbf{f} \boxtimes \mathbf{g} = 1 \quad (\text{or} \quad \mathbf{f} \circledvee \mathbf{g} = \phi). \tag{3-6}$$

As in the case for matrices [38], each solution to equation (3-6) consists of an assignment of one of the values $-\infty$ or $\phi$ to each $\mathbf{f}(\mathbf{x}_j)$.

Let $\mathbf{f} = (\mathbf{f}(\mathbf{x}_1),...,\mathbf{f}(\mathbf{x}_m))$ be a solution to equation (3-6).

**Step 2.** For each $j = 1,...,m$: if $\mathbf{f}(\mathbf{x}_j) = \phi$ then set $\mathbf{a}(\mathbf{x}_j)$ to be the value $-\left(\bigvee \mathbf{s}'_{\mathbf{x}_j}\right)$ $(1/(\left(\bigvee \mathbf{s}'_{\mathbf{x}_j}\right)$. If $\mathbf{f}(\mathbf{x}_j) = -\infty$ then $\mathbf{a}(\mathbf{x}_j)$ is given an arbitrary value such that $\mathbf{a}(\mathbf{x}_j) <$ $-\left(\bigvee \mathbf{s}'_{\mathbf{x}_j}\right)(1/(\left(\bigvee \mathbf{s}'_{\mathbf{x}_j}\right)$.

For the boolean case, we have

**Proposition 3.17.** *The solutions of equation* (3-6) *are exactly the assignments of the values* $\phi$ *or* $-\infty$ *to the variables* $\mathbf{f}(\mathbf{x}_j)$ *such that for every* $i = 1,...,m$ *there holds* $\mathbf{f}(\mathbf{x}_j) = \phi$ *for at least one* $j$ *such that* $\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j)$ *is a marked value.*

**Theorem 3.18.** *Let* $\mathrm{F}_{\pm\infty}$ *be a bounded l-group. Then the above two step procedure yields all solutions to equation* (3-7) *without repetition.*

### 3.2.3. Existence and Uniqueness

This section discusses some existence and uniqueness theorems concerning solutions to Problem (3-3).

**Theorem 3.19.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, and let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *be doubly* $\mathbf{F}$*-astic and* $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}$ *be finite. Then a necessary and sufficient condition that the equation* $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ $(\mathbf{a} \circledotdown \mathbf{t} = \mathbf{b})$ *shall have at least one solution is that for all* $\mathbf{x}_i \in \mathbf{X}$*, there exists at least one* j *such that for the template* $\mathbf{s} = \mathbf{d} \boxtimes \mathbf{t}$ $(\mathbf{s} = \mathbf{d} \circledotdown \mathbf{t})$*, where* $\mathbf{d}$ *is as defined as above,*

$$\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j) \textit{ is a marked value.}$$

We remark that the solution $\mathbf{a}(\mathbf{x}_j) = -\left(\bigvee \mathbf{s}'_{\mathbf{x}_j}\right)\ (1/(\left(\bigvee \mathbf{s}'_{\mathbf{x}_j}\right)))$ gives exactly the principal solution.

This is equivalent to

**Theorem 3.20.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *be doubly* $\mathbf{F}$*-astic, and let* $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}$ *be finite. Then a necessary and sufficient condition that the equation* $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ $(\mathbf{a} \circledotdown \mathbf{t} = \mathbf{b})$ *shall have exactly one solution is that for all* $\mathbf{x}_i \in \mathbf{X}$*, there exists at least one* j *such that*

$$\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j) \textit{ is a marked value,}$$

*and for each* $j = 1,\ldots,n$*, there exists an* i, $1 \le i \le m$ *such that* $|W'_i| = 1$.

Define a template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ to be *strictly doubly* $\phi$*-astic* if it satisfies the following two conditions.

(i) $\mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j) \le \phi$, i,j=1,...,n

(ii) for each $i = 1,\ldots,n$, there exists a unique index $j \in \{1,2,\ldots,n\}$ such that $\mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j)$ has value $\phi$.

If $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$, $|\mathbf{X}| = m$, $|\mathbf{Y}| = n$, then we say that $\mathbf{t}$ *contains a template* $\mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{W}_2})^{\mathbf{W}_1}$ if the matrix $\Psi^{-1}(\mathbf{t})$ contains the matrix $\Psi^{-1}(\mathbf{s})$ of size h x k, where $|\mathbf{W}_2| = h$, $|\mathbf{W}_1| = k$, and both $h,k \le \min(m,n)$. We say that a template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *contains an image* $\mathbf{a} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$ if $\mathbf{a} = \mathbf{t}_{\mathbf{y}}$ for some $\mathbf{y} \in \mathbf{Y}$.

**Theorem 3.21.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *be doubly* $\mathbf{F}$*-astic, and let* $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}$ *be finite. Then a necessary and sufficient condition that the equation* $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ ($\mathbf{a} \circledR \mathbf{t} = \mathbf{b}$) *shall have exactly one solution is that we can find* k *finite elements* $\alpha_1, \ldots, \alpha_k$ *such that the template* $\mathbf{d}$ *defined by*

$$\mathbf{d}_{\mathbf{y}_i}(\mathbf{x}_j) = -\mathbf{b}(\mathbf{y}_i) + \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j) + \alpha_j \quad (\text{ or } \mathbf{d}_{\mathbf{y}_i}(\mathbf{x}_j) = \mathbf{b}(\mathbf{y}_i)^{-1} * \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j) * \alpha_j$$

*is doubly* $\phi$*-astic and that* $\mathbf{d}$ *contains a strictly doubly* $\phi$*-astic template* $\mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{W}})^{\mathbf{W}}$, $|\mathbf{W}| = \mathrm{k}$.

### 3.2.4. A Linear Programming Criterion

Since one of our interests is the case where the bounded l-group is the $\mathbf{R}_{\pm\infty}$, we now show that the problem can be stated as a linear programming problem for this bounded l-group.

**Theorem 3.22.** *Let* $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *be doubly* $\mathbf{F}$*-astic and* $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}$ *be finite. Let* I *be the set of index pairs (i,j) such that* $\mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j)$ *is finite,* $1 \leq \mathrm{i} \leq \mathrm{n}$, $1 \leq \mathrm{j} \leq \mathrm{m}$. *Then a sufficient condition that the equation* $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ *be soluble is that some solution* $\{\, \xi_{ij} \mid (\mathrm{i,j}) \in \mathrm{I} \,\}$ *of the following optimization problem in the variables* $z_{ij}$, *for* $(\mathrm{i,j}) \in \mathrm{I}$:

$$\textbf{Minimize} \quad \sum_{(i,j)\in I} (\mathbf{b}(\mathbf{y}_i) - \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j))\, z_{ij}$$

$$\textbf{Subject to} \quad \left( \sum_{\substack{i=1 \\ (i,j)\in I}}^{m} z_{ij} \right) = 1,\ j = 1,\ldots,m$$

$$\textbf{and} \quad z_{ij} \geq 0,\ (i,j) \in I$$

*shall also satisfy:* $\left( \sum_{\substack{j=1 \\ (i,j)\in I}}^{m} \xi_{ij} \right) > 0,\ i = 1,\ldots,n.$

We now make a definition which will be used in the next section. Let $\mathbf{F}_{\pm\infty}$ be a belt, and let $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ be arbitrary. The *right column space of* $\mathbf{t}$ is the set of all $\mathbf{b} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$ for

which the equation

$$\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b} \quad (\text{ or } \quad \mathbf{a} \circledvee \mathbf{t} = \mathbf{b}\ )$$

is soluble for $\mathbf{a}$.

3.2.5. Linear Dependence

Linear dependence over a bounded l-group. We can consider the equation $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ (or $\mathbf{a} \circledvee \mathbf{t} = \mathbf{b}$) in another way. For the images $\mathbf{t}'_{\mathbf{x}_j}$, rewrite $\mathbf{a} \boxtimes \mathbf{t} = \mathbf{b}$ as

$$\bigvee_{j=1}^{m} \left[\mathbf{t}'_{\mathbf{x}_j} \boxtimes \mathbf{a}(\mathbf{x}_j)\right] = \mathbf{b}\,, \tag{3-7}$$

where $\mathbf{a}(\mathbf{x}_j) \in (\mathbf{F}^{\mathbf{Y}}_{\pm\infty})^{\mathbf{Y}}$ is the one-point template with target pixel value of $\mathbf{a}(\mathbf{x}_j)$. In this case, we say that $\mathbf{b}$ is a *linear combination of* $\{\mathbf{t}'_{\mathbf{x}_1}, \mathbf{t}'_{\mathbf{x}_2}, \ldots, \mathbf{t}'_{\mathbf{x}_m}\}$, or, that $\mathbf{b} \in \mathbf{F}^{\mathbf{X}}_{\pm\infty}$ is *(right) linearly dependent* on the set $\{\mathbf{t}'_{\mathbf{x}_1}, \mathbf{t}'_{\mathbf{x}_2}, \ldots, \mathbf{t}'_{\mathbf{x}_m}\}$. We can make analogous definitions for the case of $\circledvee$. While in linear algebra the concept of linear dependence provides a foundation for a theory of rank and dimension, the situation in the minimax algebra is more complicated. The notion of *strong linear independence* is introduced to give us a similar construct.

**Theorem 3.23.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group other than* $\mathbf{F}_3$. *Let* $\mathbf{X}$ *be a coordinate set such that* $|\mathbf{X}| > 2$, *and* $\mathrm{k} > 1$ *be an arbitrary integer. Then we can always find* k *finite images on* $\mathbf{X}$, *no one of which is linearly dependent on the others.*

If $\mathbf{F}_{\pm\infty} = \mathbf{F}_3$, then we can produce a dimensional anomaly.

**Theorem 3.24.** *Suppose* $\mathbf{F}_{\pm\infty} = \mathbf{F}_3$, *and let* $\mathbf{X}$ *be a coordinate set such that* $|\mathbf{X}| = \mathrm{m} > 2$. *Then we can always find at least* $(\mathrm{m}^2 - \mathrm{m})$ *images on* $\mathbf{X}$, *no one of which is linearly dependent on the others.*

Since every bounded l-group contains a copy of $\mathbf{F}_3$, the dimensional anomaly in Theorem 3.24 extends to any arbitrary bounded l-group.

Let $|\mathbf{X}| = m$, $|\mathbf{Y}| = n$, and $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$ where $\mathbf{F}$ is an arbitrary bounded l-group. We would like to define the rank of $\mathbf{t}$ in terms of linear independence, and to be equal to the number of linearly independent images $\mathbf{t}'_{\mathbf{x}_j}$ of $\mathbf{t}$. Suppose we were to define linear independence as the negation of linear dependence, that is, a set of k images on $\mathbf{X}$ ( $\mathbf{a}_1, \ldots, \mathbf{a}_k$ ) is linear independent if and only if no one of the $\mathbf{a}_i$ is linearly dependent on any subset of the others. Then applying Theorem 3.23 for $|\mathbf{X}| = n$ and $k \geq n$, we could find k finite images which are linearly independent. If we defined rank as the number of linearly independent images $\mathbf{t}_{\mathbf{y}}$ of $\mathbf{t}$, then every template would have rank $k \geq n$, which is not a useful definition in this context.

Strong linear independence. As for the matrix algebra, we define the concept of strong linear independence [38].

Let $\mathbf{F}_{\pm\infty}$ be a bounded l-group and let $\mathbf{a}(1),\ldots,\mathbf{a}(k) \in \mathbf{F}^{\mathbf{X}}_{\pm\infty}$, $k \geq 1$. We say that the set $\{\mathbf{a}(1),\ldots,\mathbf{a}(k)\}$ is *strongly linearly independent*, or simply SLI, if there is at least one finite image $\mathbf{b} \in \mathbf{F}^{\mathbf{X}}$ which has a unique expression of the form

$$\mathbf{b} = \bigvee_{p=1}^{h} \mathbf{a}(j_p) \boxtimes \lambda_{j_p} \quad \left(\text{or} \quad \mathbf{b} = \bigvee_{p=1}^{h} \mathbf{a}(j_p) \circledvee \lambda_{j_p}\right) \tag{3-8}$$

with $\lambda_{j_r} \in \mathbf{F}$, $p = 1,\ldots,h$, $1 \leq j_p \leq k$, $p = 1,\ldots,h$, and $j_p < j_q$ if $p < q$.

If $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ is a set of k images where each $\mathbf{a}_i \in \mathbf{F}^{\mathbf{Y}}_{\pm\infty}$, $|\mathbf{Y}| = n$, then we define the *template based on the set* $\mathcal{A}$ in the following way. For the integer k, we find a coordinate set $\mathbf{W}$ which has k pixel locations, that is, $|\mathbf{W}| = k$. To this end, choose a positive integer p such that $k = p \cdot q + r$, where $r < p$ (by the division algorithm for integers). Let $\mathbf{W}$ denote the set $\{(i,j) : 0 \leq i \leq p-1,\ 0 \leq j \leq q-1\} \cup \{(-1,j) : 0 \leq j \leq r-1\}$, which is a subset of $\mathbf{Z}^2$ that is almost rectangular. There is an additional row in the fourth quadrant

corresponding to the r left-over pixel locations that don't quite make a full row. Of course, there are other selections that can be made for $\mathbf{W}$. Define the *template* $\mathbf{t}$ *based on* $\mathcal{A}$ by $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{W}})^{\mathbf{Y}}$, where

$$\mathbf{t}'_{\mathrm{w}_i} = \mathbf{a}_i, \; i = 1,\ldots,k.$$

To clarify notation, we will denote the template based on the set $\mathcal{A} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_k\}$ by $\mathbf{t} = \mathrm{B}(\mathcal{A})$. Hence, if $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, then for $\mathcal{A} = \{\mathbf{t}'_{\mathrm{x}_1}, \mathbf{t}'_{\mathrm{x}_2}, \ldots, \mathbf{t}'_{\mathrm{x}_m}\}$, we have $\mathrm{B}(\mathcal{A}) = \mathbf{t}$. If $\mathcal{D} = \{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h\}$ is a set of h $\mathbf{F}$ valued images on $\mathbf{X}$, we denote the *right column space* of $\mathrm{B}(\mathcal{D})$ by $< \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h >$. Thus, for $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{Y}}$, $< \mathbf{t}'_{\mathrm{x}_1}, \mathbf{t}'_{\mathrm{x}_2}, \ldots, \mathbf{t}'_{\mathrm{x}_m} >$ is the right column space of $\mathbf{t}$. The set $< \mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h >$ is also called the *space generated by the set* $\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_h\}$.

**Lemma 3.25.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group with group* $\mathbf{F}$. *Let* $\mathbf{c}_1, \ldots, \mathbf{c}_k, \mathbf{b} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$, $k \geq 1$ *be such that* $\mathbf{b}$ *is finite and has a unique expression of the form* (3-8). *Then* $h = k$; $j_1 = 1, \ldots, j_h = k$; $\lambda_{j_p} \in \mathbf{F}$, $p = 1,\ldots,h$; *and* $\mathbf{t}$ *is doubly* $\mathbf{F}$*-astic, where* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *is the template based on the set* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_k\}$. *Here,* $|\mathbf{Y}| = k$.

We also have

**Corollary 3.26**. *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group and let* $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$ *for an integer* $n \geq 1$. *Then* $\{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$ *is* SLI *if and only if there exists a finite image* $\mathbf{b} \in \mathbf{F}^{\mathbf{X}}$ *such that the equation* $\mathbf{a} \boxed{\vee} \mathbf{t} = \mathbf{b}$ $(\mathbf{a} \circledvee \mathbf{t} = \mathbf{b})$ *is uniquely soluble for* $\mathbf{a}$, *where* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *is the template based on the set* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_n\}$, $\mathbf{t} = \mathrm{B}(\mathcal{C})$, $|\mathbf{Y}| = n$.

We can now define linear independence. Let $\mathbf{F}_{\pm\infty}$ be a given belt. Then *linear independence* is the negation of linear dependence: $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$ are linearly independent when no one of them is linearly dependent on the others. How is linear dependence related to strong linear independence?

**Theorem 3.27.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, and* $\mathbf{c}_1, \ldots, \mathbf{c}_n \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$. *For* $\mathbf{c}_1, \ldots, \mathbf{c}_k$ *to be linearly independent it is sufficient, but not necessary, that* $\mathbf{c}_1, \ldots, \mathbf{c}_k$ *be* SLI.

We may call the above definition of SLI *right SLI.* If, in the definition of SLI, we were to multiply by the scalars $\lambda_j$'s from the left, we define the concept of *left SLI.* If formula (3–8) is replaced by

$$\mathbf{b} = \bigwedge_{p=1}^{h} \mathbf{a}(j_p) \boxtimes \lambda_{j_p} \quad (\text{ or } \mathbf{b} = \bigwedge_{p=1}^{h} \mathbf{a}(j_p) \circledast \lambda_{j_p})$$

then we have the concept of *right dual SLI.* We define in an analogous way the concept of left dual SLI.

## 3.3. Rank of Templates

Template rank over a bounded l-group. Let $\mathbf{F}_{\pm\infty}$ be a bounded l-group and $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ be arbitrary. We call the template $\mathbf{t}$ *(right)* or *left column regular* if the set of images $\{\mathbf{t}'_{\mathbf{x}}\}_{\mathbf{x}\in\mathbf{X}}$ are (right) or left SLI, respectively. We say $\mathbf{t}$ is *right* or *left row regular* if the template $\mathbf{t}'$ is right or left column regular, respectively.

Now suppose that $\mathbf{F}_{\pm\infty}$ is a bounded l-group and $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$. Suppose r is the maximum number of images $\mathbf{t}'_{\mathbf{x}}$ of $\mathbf{t}$ that are SLI. In this case we say that $\mathbf{t}$ has *column rank* equal to r. The *row rank* of $\mathbf{t}$ is the column rank of $\mathbf{t}'$. For a template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$, we say that $\mathbf{t}$ has *$\phi$-astic rank* equal to $r \in \mathbf{Z}^+$ if the following is true for $k = r$ but not for $k > r$:

> Let $\mathbf{W}$ be a coordinate set, $|\mathbf{W}| = k \leq \min(m,n)$. There exist $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$ and $\mathbf{b} \in \mathbf{F}^{\mathbf{Y}}$, both finite, such that the template $\mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ is doubly $\phi$-astic and $\mathbf{s}$ contains a strictly doubly $\phi$-astic template $\mathbf{u} \in (\mathbf{F}_{\pm\infty}^{\mathbf{W}})^{\mathbf{W}}$, where
>
> $$\mathbf{s}_{\mathbf{y}_i}(\mathbf{x}_j) = \mathbf{b}(\mathbf{y}_i) + \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j) + \mathbf{a}(\mathbf{x}_j), \; \forall \; i = 1,\ldots,n \text{ and } j = 1,\ldots,m$$
>
> if $\mathbf{F} = \mathbf{R}$, and

$$s_{y_i}(x_j) \;=\; b(y_i) * t_{y_i}(x_j) * a(x_j)\,,\ \forall\ i = 1,\ldots,n \text{ and } j = 1,\ldots,m$$

if $F = R^+$.

**Lemma 3.28.** *Let* $F_{\pm\infty}$ *be a bounded l-group with group* $F \in \{R, R^+\}$*, and suppose that* $t \in (F^X_{\pm\infty})^Y$ *has* $\phi$*-astic rank equal to* r. *Then* t *is doubly* F*-astic and* t′ *contains a set of at least* r *images,* $t'_{x_{j_k}}$*,* k=1,...,r*, which are* SLI.

**Lemma 3.29.** *Let* $F \in \{R, R^+\}$*, and suppose that* $t \in (F^X_{\pm\infty})^Y$ *is doubly* F*-astic and consists of a set of* r *images which are* SLI. *Then* t *has* $\phi$*-astic rank equal to at least* r.

Accordingly, we have

**Theorem 3.30.** *Let* $F \in \{R, R^+\}$*, and suppose that* $t \in (F^X_{\pm\infty})^Y$ *is doubly* F*-astic. Then the following statements are all equivalent:*

(i) **t** has $\phi$-astic rank equal to r
(ii) **t** has right column rank equal to r
(iii) **t** has left row rank equal to r
(iv) $t^*$ has dual right column rank equal to r
(v) $t^*$ has dual left row rank equal to r .

If **t** is doubly **F**-astic, then we can apply Theorem 3.30 and simply use the term *rank* of **t**, for ranks (i) to (iii), and the term *dual rank* of **t** for ranks (iv) and (v). If the bounded l-group $F_{\pm\infty}$ is commutative, as in both our cases, we have the following

**Corollary 3.31**. *Let* $F \in \{R, R^+\}$*, and let* $t \in (F^X_{\pm\infty})^Y$ *be doubly* F*-astic. Then the following statements are all equivalent:*

(i) **t** has left column rank equal to r
(ii) **t** has right row rank equal to r
(iii) $t^*$ has dual left column rank equal to r
(iv) $t^*$ has dual right row rank equal to r .

3.3.1. Existence of Rank and Relation to SLI

We now discuss the existence of the rank of a template and the relationship of rank to SLI.

**Theorem 3.32.** *Let* $\mathbf{F} \in \{\mathbf{R}, \mathbf{R}^+\}$, *and let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$. *Then there is an integer* r *such that* $\mathbf{t}$ *has* $\phi$*-astic rank* r, *if and only if* $\mathbf{t}$ *is doubly* $\mathbf{F}$*-astic. In this case,* r *satisfies* $1 \leq r \leq \min(m,n)$, *where* $m = |\mathbf{X}|$, $n = |\mathbf{Y}|$.

We now have the tools to show that the previous dimension anomalies are avoided in context of strong linear independence.

**Theorem 3.33.** *Let* $\mathbf{F} \in \{\mathbf{R}, \mathbf{R}^+\}$, $\mathbf{X}$ *an arbitrary non-empty, finite coordinate set with* $|\mathbf{X}| = m$. *Then for each integer* n, $1 \leq n \leq m$, *we can find* n *images on* $\mathbf{X}$, $\mathbf{a}_j \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$, $j = 1,\ldots,n$ *which are* SLI. *This is impossible for* $n > m$.

3.3.2. Permanents and Inverses

As in linear algebra, if $\mathbf{t}$ is a matrix all of whose eigenvalues satisfy $|\lambda| < 1$, then the expression

$$(\mathbf{e} - \mathbf{t})^{-1} = \mathbf{e} + \mathbf{t} + \mathbf{t}^2 + \cdots$$

is valid. We state an analogous case in the image algebra.

For a bounded l-group $\mathbf{F}_{\pm\infty}$, a template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ is called *increasing* if

$$\mathbf{a} \boxtimes \mathbf{t} \geq \mathbf{a} \ \text{ for all } \mathbf{a} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}, \text{ and } \quad \mathbf{s} \boxtimes \mathbf{t} \geq \mathbf{s} \ \text{ for all } \mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}},$$

where $\mathbf{Y}$ is any arbitrary coordinate set.

We have

**Lemma 3.34.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, and let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$. *Then* $\mathbf{t}$ *is increasing if and only if* $\mathbf{t}_{\mathbf{x}}(\mathbf{x}) \geq \phi \ \forall\, \mathbf{x} \in \mathbf{X}$.

Let $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ be a template, $|\mathbf{X}| = m$. We define the *permanent of* $\mathbf{t}$ to be the scalar $\mathrm{Perm}(\mathbf{t}) \in \mathbf{R}_{\pm\infty}$ given by

$$\mathrm{Perm}(\mathbf{t}) = \bigvee_{\sigma} \Big( \sum_{i=1}^{m} \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_{\sigma(i)}) \Big),$$

where the maximum is taken over all permutations $\sigma$ in the symmetric group $S_m$ of order $m!$.

For the bounded l-group $\mathbf{R}_{\pm\infty}^{+}$, let $\mathbf{t} \in ((\mathbf{R}_{\pm\infty}^{+})^{\mathbf{X}})^{\mathbf{X}}$ be a template, $|\mathbf{X}| = m$. We define the *permanent of* $\mathbf{t}$ to be the scalar $\mathrm{Perm}(\mathbf{t}) \in \mathbf{R}_{\pm\infty}^{+}$ given by

$$\mathrm{Perm}(\mathbf{t}) = \bigvee_{\sigma} \Big( \prod_{i=1}^{m} \mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_{\sigma(i)}) \Big),$$

where again the maximum is taken over all permutations $\sigma$ in the symmetric group $S_m$.

The *adjugate* template of $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ is the template $\mathrm{Adj}(\mathbf{t})$ defined by

$$[\mathrm{Adj}(\mathbf{t})]_{\mathbf{y}_i}(\mathbf{x}_j) = \mathrm{Cofactor}[\mathbf{t}]_{\mathbf{x}_j}(\mathbf{y}_i)$$

where $\mathrm{Cofactor}[\mathbf{t}]_{\mathbf{x}_j}(\mathbf{y}_i)$ is the permanent of the template $\mathbf{s}$ defined by

$$\mathbf{s}_{\mathbf{y}_k}(\mathbf{x}_h) = \mathbf{t}_{\mathbf{y}_k}(\mathbf{x}_h)$$

$$h = 1,\ldots,j-1,j+1,\ldots,m \text{ and } k = 1,\ldots,i-1,i+1,\ldots,m\,.$$

Here, $\mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{W}})^{\mathbf{W}}$, where $|\mathbf{W}| = m-1$. For $m = 1$, we define $\mathrm{Adj}(\mathbf{t}) = \phi$.

### 3.3.3. Graph Theory

We now present some graph theoretic tools which will be used later.

A *digraph* or *directed graph* is a pair $D = \{V,E\}$ where $V$ is a finite set of vertices $\{1,\ldots,n\}$ and $E \subset V \times V$. The set $E$ is called the set of *edges* of $D$. An edge $(i,j)$ is *directed* from $i$ to $j$, and can be represented by a vector with tail at node $i$ and head at node $j$.

A *graph* is a pair $G = \{V,E\}$ where $V$ is a finite set of vertices $\{1,\ldots,n\}$ and $E \subset V \times V$ such that $(i,j) \in E$ if and only if $(j,i) \in E$.

A u-v *path* in a digraph or graph is a finite sequence of vertices $u = y_0, y_1, ..., y_m = v$ such that $(y_j, y_{j+1}) \in E$ for all $j = 0,...,m\text{-}1$. A *circuit* is a path with the property that $y_0 = y_m$. A *simple path* $y_0, y_1, ..., y_m$ is a path with distinct vertices except possibly for $y_0$ and $y_m$. A *simple circuit* is a circuit which is a simple path.

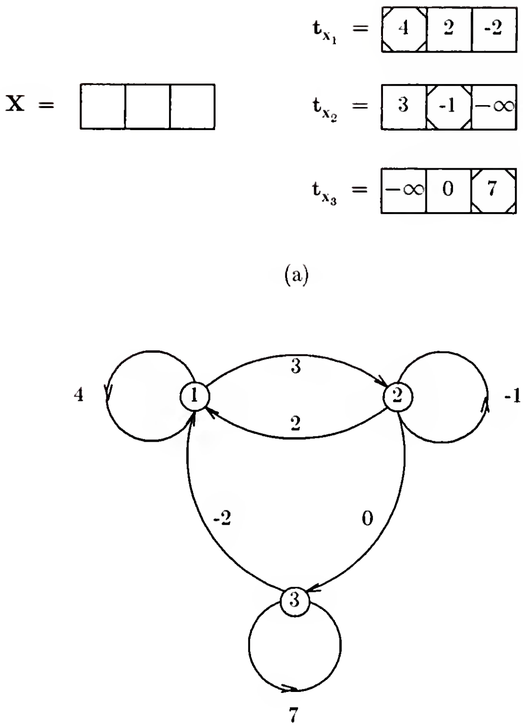
A *weighted digraph* (*graph*) is a digraph (graph) to which every edge (i,j) is uniquely assigned a value in $\mathbf{F}_{\pm\infty}$. We denote the weight of the edge (i,j) by t(i,j) or $t_{ij}$. Note that the value $t_{ij}$ is not necessarily equal to the value $t_{ji}$.

We remark that if $G = \{V,E\}$ is a graph then if there exists a u-v path, there exists a v-u path.

With each path (circuit ) $\sigma = y_0, y_1, ..., y_m$ of a weighted graph G, there is an associated *path (circuit) product* $p(\sigma)$, defined by

$$t_{y_0,y_1} \times t_{y_1,y_2} \times \cdots \times t_{y_{m-1},y_m}.$$

For each template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}}_{-\infty})^{\mathbf{X}}$ where $|\mathbf{X}| = n$, we can associate a weighted graph $\Delta(\mathbf{t})$ in the following way. The associated graph $\Delta(\mathbf{t})$ is the weighted graph $G = (V,E)$, where $V = \{1,2,...,n\}$, and whose weights are $\mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i)$, for the pair (i,j) such that $\mathbf{x}_i \in \mathcal{S}_{-\infty}(\mathbf{t}_{\mathbf{x}_j})$. The pair (i,j) is then considered an edge. If $\mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) = -\infty$, then we can extend E to all of $V \times V$ by stating that $(i,j) \in E$ with null weight $-\infty$. An example of a template $\mathbf{t}$ and its associated weighted graph $\Delta(\mathbf{t})$ is given below in Figure 7. We have omitted listing the values of $-\infty$ on $\Delta(\mathbf{t})$. Here, $|\mathbf{X}| = 3$.




(b)

**Figure 7.** A Template and its Associated Graph.
(a) A Template **t**; (b) Associated Graph $\Delta(\mathbf{t})$.

For the belt $\mathbf{F}_{-\infty}$, the correspondence is one-one. We note this in the next theorem.

**Lemma 2.48**. *Let* $\mathbf{F}_{-\infty}$ *be a belt, where* $+\infty \notin \mathbf{F}$. *Let* $\alpha : (\mathbf{F}^{\mathbf{X}}_{-\infty})^{\mathbf{X}} \rightarrow \mathcal{G} = \{\, G : G$ *is a weighted graph with n nodes* $\}$ *be defined by* $\alpha(\mathbf{t}) = \Delta(\mathbf{t})$. *Then* $\alpha$ *is one-one and onto.*

**Proof:** Suppose $\alpha(\mathbf{t}) = \alpha(\mathbf{s})$. Let $\{\, \mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) \,\}$ be the weights for $\Delta(\mathbf{t})$ and $\{\, \mathbf{s}_{\mathbf{x}_j}(\mathbf{x}_i) \,\}$ be the weights for $\Delta(\mathbf{s})$. By definition, $\mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) = \mathbf{s}_{\mathbf{x}_j}(\mathbf{x}_i)$ for all i,j, and hence $\mathbf{t} = \mathbf{s}$.

Now suppose that $G = (V,E)$ is a weighted graph with weights $\{ w_{ij} \}$. Define $\mathbf{t} \in (\mathbf{F}_{-\infty}^{X})^{X}$ by $\mathbf{t}_{x_j}(\mathbf{x}_i) = w_{ij}$, if $(i,j) \in E$, and $\mathbf{t}_{x_j}(\mathbf{x}_i) = -\infty$ otherwise. Then $\alpha(\mathbf{t}) = G$.

Q.E.D.

Let $\mathbf{t} \in (\mathbf{F}_{-\infty}^{X})^{X}$. If for each circuit $\sigma$ in $\Delta(\mathbf{t})$ we have $p(\sigma) \leq \phi$ and there exists at least one circuit $\sigma$ such that $p(\sigma) = \phi$, then we call $\mathbf{t}$ a *definite* template.

**Lemma 3.35.** *A template* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{X})^{X}$ *is definite if and only if for all simple circuits* $\sigma \in \Delta(\mathbf{t})$, $p(\sigma) \leq \phi$ *and there exists at least one such simple circuit* $\sigma$ *such that* $p(\sigma) = \phi$.

**Theorem 3.36.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{X})^{X}$ *be either row-$\phi$-astic or column-$\phi$-astic. Then* $\mathbf{t}$ *is definite.*

**Theorem 3.37.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{X})^{X}$. *If* $\mathbf{t}$ *is definite then so is* $\mathbf{t}^r$, *for any integer* $r \geq 0$.

Let $\mathbf{t} \in (\mathbf{F}_{-\infty}^{X})^{X}$ where $|\mathbf{X}| = n$. The *metric template generated by* $\mathbf{t}$ is

$$\Gamma(\mathbf{t}) = \mathbf{t} \vee \mathbf{t}^2 \vee \cdots \vee \mathbf{t}^n .$$

The *dual metric template* is

$$\Gamma^*(\mathbf{t}) = \mathbf{t}^* \wedge (\mathbf{t}^2)^* \wedge \cdots \wedge (\mathbf{t}^n)^* .$$

The name *metric* originates from the application of the minimax algebra to transportation networks. If for the bounded l-group $\mathbf{R}_{\pm\infty}$ the value $\mathbf{t}_{x_j}(\mathbf{x}_i)$ represents the direct distance from node i to node j of a transportation network with $\mathbf{t}_{x_j}(\mathbf{x}_i) = +\infty$ if there is no direct route, then $(\Gamma(\mathbf{t}))^*$ represents the shortest distance matrix, that is, $((\Gamma(\mathbf{t}))^*)_{x_j}(\mathbf{x}_i)$ is the shortest path possible from node i to node j of all possible paths. A description of a transportation problem concerning shortest paths is discussed in Cuninghame-Green's book [38].

Theorems 3.38 through 3.40 are used to prove Theorem 3.41.

**Lemma 3.38.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$. *Then*

$$\Gamma(\mathbf{t}) = (\mathbf{1} \vee \mathbf{t})^{n-1} \boxtimes \mathbf{t}\,.$$

**Lemma 3.39.** $(\mathbf{t} \vee \mathbf{1})^{n-1} = \mathbf{1} \vee \mathbf{t} \vee \cdots \vee \mathbf{t}^{n-1}$, $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$.

**Theorem 3.40.** *Let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be definite. Then*

$$\mathbf{t}^r \leq \Gamma(\mathbf{t})\,,\ r = 1,2,\ldots$$

**Theorem 3.41.** *Let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be definite. Then*

$$[\,\Gamma(\mathbf{t})\,]^r \leq \Gamma(\mathbf{t})\,,\ r = 1,2,\ldots$$

$$\Gamma(\mathbf{t}) = (\mathbf{1} \vee \mathbf{t})^r \boxtimes \mathbf{t}\ , r = 1,2,\ldots,n-1\,.$$

Using the Adjugate of a template, we have

**Theorem 3.42** [52]. *Let* $\mathbf{F}_{\pm\infty}$ *be a commutative bounded l-group and* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be definite and increasing. Then* $\mathrm{Adj}(\mathbf{t}) = \Gamma(\mathbf{t})$.

Now we define the *inverse* of a template. For $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$, we define

$$\mathrm{Inv}(\mathbf{t}) = (\mathrm{Perm}(\mathbf{t}))^{-1} \boxtimes \mathrm{Adj}(\mathbf{t}) \quad (\text{ or } \quad \mathrm{Inv}(\mathbf{t}) = (\mathrm{Perm}(\mathbf{t}))^{-1} \circledvee \mathrm{Adj}(\mathbf{t})\ )$$

by direct analogy in elementary linear algebra.

We note that the template $\mathrm{Inv}(\mathbf{t})$ is not necessarily invertible in the sense that $\mathrm{Inv}(\mathbf{t}) \boxtimes \mathbf{t} = \mathbf{1}$, for example.

### 3.3.4. Invertibility

In order to define an *invertible* template, that is, a template $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ that has the property that there exists a unique template $\mathbf{s}$ satisfying $\mathbf{t} \boxtimes \mathbf{s} = \mathbf{s} \boxtimes \mathbf{t} = \mathbf{1}$ ($\mathbf{t} \circledvee \mathbf{s} = \mathbf{s} \circledvee \mathbf{t} = \mathbf{1}$), we need to introduce the concept of *equivalent templates.*

Let $\mathbf{F}_{\pm\infty}$ be a subbelt of $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^{+}_{\pm\infty}$. A template $\mathbf{p} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}$ is said to be *invertible* if there exists a template $\mathbf{q} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}$ such that $\mathbf{p} \boxtimes \mathbf{q} = \mathbf{q} \boxtimes \mathbf{p} = \mathbf{l}$ ($\mathbf{p} \circledvee \mathbf{q} = \mathbf{q} \circledvee \mathbf{p} = \mathbf{l}$).

These templates can be described in close detail. Let us define a *strictly doubly* $\mathbf{F}$*-astic template* over a bounded l-group $\mathbf{F}_{\pm\infty}$ to be an element $\mathbf{t}$ of $(\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}$ satisfying

(i) $\mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_j) < +\infty$, i,j = 1,...,n

(ii) for each index i there exists a unique index $j_i \in \{1,2,...,m\}$ such that $\mathbf{t}_{\mathbf{y}_i}(\mathbf{x}_{j_i})$ is finite.

**Theorem 3.43.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group with group* $\mathbf{F}$ *and let* $\mathbf{p} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}$ *be given. Then* $\mathbf{p}$ *is invertible if and only if* $\mathbf{p}$ *is strictly doubly* $\mathbf{F}$*-astic.*

As is usual, if $\mathbf{p}$ is invertible, then the template $\mathbf{q}$ above is written as $\mathbf{p}^{-1}$.

The intersection of the set of strictly doubly $\phi$-astic templates and the set of strictly doubly $\mathbf{F}$-astic templates we call the *permutation templates*. It is not difficult to show

**Proposition 3.44.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group. Then the set of invertible templates from* $\mathbf{X}$ *to* $\mathbf{X}$*, where* $|\mathbf{X}| = m$*, form a group under the multiplication* $\boxtimes$ ($\circledvee$)*, containing* $\mathbf{l}$ *as the identity element and having the permutation templates as a subgroup isomorphic to the symmetric group* $S_m$ *on* m *letters.*

Pre- or post-multiplication of a template $\mathbf{t}$ by a permutation template $\mathbf{p}$ will permute the images $\mathbf{t}'_{\mathbf{x}}$ or the images $\mathbf{t}_{\mathbf{y}}$ of $\mathbf{t}$, respectively, and these permutation templates play a role exactly like their counterparts in linear algebra.

### 3.3.5. Equivalence of Templates

Let $\mathbf{F}_{\pm\infty}$ be a bounded l-group, and let $\mathbf{t}, \mathbf{s} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$ be given. We say that $\mathbf{t}$ and $\mathbf{s}$ are *equivalent*, written $\mathbf{t} \equiv \mathbf{s}$, if there exist invertible templates $\mathbf{p} \in (\mathbf{F}^{\mathbf{Y}}_{\pm\infty})^{\mathbf{Y}}$ and $\mathbf{q} \in (\mathbf{F}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}$ such that $\mathbf{p} \boxtimes \mathbf{t} \boxtimes \mathbf{q} = \mathbf{s}$ ($\mathbf{p} \circledvee \mathbf{t} \circledvee \mathbf{q} = \mathbf{s}$).

Now we define *elementary templates.* An *elementary template* $\mathbf{p} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ over a bounded l-group with group $\mathbf{F}$ is one of the following:

(i) a permutation template
(ii) a diagonal template of the form diag$(\phi, ..., \phi, \alpha, \phi, \ldots, \phi)$, where $\alpha \in \mathbf{F}$.

Elementary templates correspond to matrices which perform elementary operations on matrices [38]. A permutation template

1. permutes the images $\mathbf{t}'_{\mathbf{x}}$ of $\mathbf{t}$; or
2. permutes the images $\mathbf{t}_{\mathbf{y}}$ of $\mathbf{t}$,

depending on whether the multiplication is from the left or right, respectively. Diagonal templates of the type listed in (ii) above have the effect of multiplying some image $\mathbf{t}'_{\mathbf{x}}$ of $\mathbf{t}$ by a finite constant $\alpha$, or multiplying some image $\mathbf{t}_{\mathbf{y}}$ of $\mathbf{t}$ by a finite constant $\alpha$, depending on whether the multiplication of $\mathbf{t}$ is from the left or right, respectively.

**Lemma 3.45.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, and let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be given coordinate sets,* $|\mathbf{X}| = m, |\mathbf{Y}| = n$. *Then the relation of equivalence is an equivalence relation on* $(\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$. *If* $\mathbf{t}, \mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$, *then* $\mathbf{t} \equiv \mathbf{s}$ *if and only if there is a sequence of templates* $\mathbf{u}_0, \mathbf{u}_1, \ldots, \mathbf{u}_j$ *such that* $\mathbf{u}_0 = \mathbf{t}$ *and* $\mathbf{u}_j = \mathbf{s}$, *and* $\mathbf{u}_p$ *is obtained by an elementary operation on* $\mathbf{u}_{p-1}$, $p = 1, \ldots, j$.

Permutation and diagonal templates of this form will play an important role in the discussion on local template decompositions, as well as the following theorem.

**Lemma 3.46.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group with group* $\mathbf{F}$ *and let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *be given. If a given image of* $\mathbf{t}'$ *(or* $\mathbf{t}$*) is* $\mathbf{F}$*-astic then* $\mathbf{t}$ *is equivalent to a template in which that image of* $\mathbf{t}'$ *(or* $\mathbf{t}$*) is* $\phi$*-astic and all other images in* $\mathbf{t}'$ *(or* $\mathbf{t}$*) are identical with the corresponding image in* $\mathbf{t}'$ *(or* $\mathbf{t}$*). Hence if* $\mathbf{t}$ *is (row-, column-, or doubly)* $\mathbf{F}$*-astic then* $\mathbf{t}$ *is equivalent to a template which is (respectively row-, column-, or doubly)* $\phi$*-astic.*

Equivalence and rank. The following are results which show the relation between equivalence and rank.

**Proposition 3.47.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group, and let* $\mathbf{t}, \mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$. *Then* $\mathbf{t}$ *has* $\phi$-*astic rank equal to* $r$ *if and only if the following statement is true for* $j = r$ *but not for* $j > r$:

> $\mathbf{t}$ is equivalent to a doubly $\phi$-*astic template* $\mathbf{d}$ which contains a strictly doubly $\phi$-astic template $\mathbf{u} \in (\mathbf{F}_{\pm\infty}^{\mathbf{W}})^{\mathbf{W}}$, where $|\mathbf{W}| = j$.

**Corollary 3.48.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group with group* $\mathbf{F}$ *and let* $\mathbf{t}, \mathbf{s} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$ *be equivalent. Then if either* $\mathbf{t}$ *or* $\mathbf{s}$ *has a rank, then so does the other, and the ranks are equal.*

## 3.4. The Eigenproblem in the Image Algebra

Using the isomorphism, we can discuss the eigenproblem which is presented in its matrix form [38] in context of the image algebra. In this section we present the eigenproblem and solution in image algebra notation.

### 3.4.1. The Statement in Image Algebra

Unless otherwise stated, we assume that $\mathbf{F}$ is a subbelt of either $\mathbf{R}$ or $\mathbf{R}^+$, and let $\mathbf{F}_{\pm\infty}$, $\mathbf{F}_{-\infty}$, and $\mathbf{F}_{+\infty}$ have their usual meanings. The coordinate sets $\mathbf{X}$ and $\mathbf{Y}$ are assumed to be non-empty, finite arrays, with $|\mathbf{X}| = m$ and $|\mathbf{Y}| = n$.

Let $\lambda \in \mathbf{F}_{\pm\infty}$. Let $\boldsymbol{\lambda} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ be the one-point template defined in the usual way by

$$\boldsymbol{\lambda}_{\mathbf{y}}(\mathbf{x}) = \begin{cases} \lambda & \mathbf{x} = \mathbf{y} \\ -\infty & \text{otherwise} \end{cases}.$$

Suppose $\mathbf{F}$ is a subbelt of $\mathbf{R}$, and $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$. Then the eigenproblem is to find $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$ and $\lambda \in \mathbf{F}_{\pm\infty}$ such that

$$\mathbf{a} \boxtimes \mathbf{t} = \mathbf{a} \boxtimes \lambda .$$

Similarly for the operation $\textcircled{\wedge}$, we need find $\mathbf{a} \in \mathbf{F}^X$ and $\lambda \in \mathbf{F}_{\pm\infty}$ such that

$$\mathbf{a} \textcircled{\wedge} \mathbf{t} = \mathbf{a} \textcircled{\wedge} \lambda .$$

For either belt, if such $\mathbf{a}$ and $\lambda$ exist, then $\mathbf{a}$ is called an *eigenimage* of $\mathbf{t}$, and $\lambda$ a corresponding *eigenvalue.* The eigenproblem is called *finitely soluble* if both $\mathbf{a}$ and $\lambda$ are finite.

As mentioned before, all results of this section are applicable for $\mathbf{F}$ a subbelt of with $\mathbf{R}$ or $\mathbf{R}^+$. Hence, to avoid stating all results for both belts separately, we will state the results for $\boxtimes$ with the understanding that in all theorems, definitions, etc. in this section of Chapter 3, with the exception of Theorem 3.57, $\boxtimes$ can be replaced by $\textcircled{\wedge}$ everywhere and the theorems and results will still hold.

**Theorem 3.49.** *Let* $\mathbf{t} \in (\mathbf{F}^X)^Y$. *Then there exist* $\mathbf{s} \in (\mathbf{F}^Y_{\pm\infty})^Y$ *such that if* $\mathbf{b}$ *is in the column space of* $\mathbf{t}$, *then* $\mathbf{b}$ *is an eigenimage of* $\mathbf{s}$ *with corresponding eigenvalue* $\phi$. *Here,* $\mathbf{s} = \mathbf{t}^* \boxtimes \mathbf{t} \in (\mathbf{F}^X_{-\infty})^X$. *Hence,* $\mathbf{b} \boxtimes \mathbf{t} = \mathbf{b} \boxtimes \mathbf{1} = \mathbf{b}$.

**Theorem 3.50.** *Let* $\mathbf{t} \in (\mathbf{F}^X_{-\infty})^X$. *If the eigenproblem for* $\mathbf{t}$ *is finitely soluble,* $\mathbf{t}$ *must be row-*$\mathbf{F}$*-astic. In particular, if* $\mathbf{t}$ *is row-*$\phi$*-astic, then the eigenproblem for* $\mathbf{t}$ *is finitely soluble, in which case* $\lambda = \phi$.

Let $\mathbf{t} \in (\mathbf{F}^X_{\pm\infty})^X$ be definite. We know that $\Delta(\mathbf{t})$ has at least one circuit $\sigma$ such that $p(\sigma) = \phi$. An *eigennode* of $\Delta(\mathbf{t})$ is any node on such a circuit. Two eigennodes are *equivalent* if they are both on any one such circuit.

**Lemma 3.51.** *Let* $\mathbf{t} \in (\mathbf{F}^X_{\pm\infty})^X$ *be definite. Then* $\Gamma(\mathbf{t})$ *is definite, and if* $j$ *is an eigennode of* $\Delta(\mathbf{t})$, *then*

$$(\Gamma(\mathbf{t}))_{\mathbf{x}_j}(\mathbf{x}_j) = \phi .$$

*Conversely, if* $(\Gamma(\mathbf{t}))_{\mathbf{x}_j}(\mathbf{x}_j) = \phi$ *for some* $\mathbf{x}_j \in \mathbf{X}$, *then* $j$ *is an eigennode of* $\Delta(\mathbf{t})$.

**Lemma 3.52.** *Let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *be definite. If* j *is an eigennode of* $\Delta(\mathbf{t})$ *then*

$$\mathbf{a}^{\mathrm{j}} \boxtimes \mathbf{t} \;=\; \mathbf{a}^{\mathrm{j}} \boxtimes \mathbf{1} \;=\; \mathbf{a}^{\mathrm{j}}$$

*where* $\mathbf{a}^{\mathrm{j}}$ *is the image* $[\,\Gamma(\mathbf{t})\,]'_{\mathrm{x_j}}$.

Thus, images $[\,\Gamma(\mathbf{t})\,]'_{\mathrm{x_j}}$ where j is an eigennode give us eigenimages for the template $\mathbf{t}$, with corresponding eigenvalue $\phi$. For a given $\mathbf{t}$, the set of all such images are called the *fundamental eigenimages* for $\mathbf{t}$. Just as in the case for matrices, two fundamental eigenimages are called *equivalent* if nodes j and h are equivalent, and otherwise the eigenimages are non-equivalent.

**Theorem 3.53.** *Let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *be definite. If* $\mathbf{a}^{\mathrm{j}}$, $\mathbf{a}^{\mathrm{k}} \in \mathbf{F}_{\pm\infty}^{\mathrm{X}}$ *are fundamental eigenvectors of* $\mathbf{t}$ *corresponding to equivalent eigennodes* j *and* k, *respectively, then*

$$\mathbf{a}^{\mathrm{j}} \;=\; \mathbf{a}^{\mathrm{k}} \boxtimes \boldsymbol{\alpha},$$

*where* $\alpha \in \mathbf{F}$, *and* $\boldsymbol{\alpha} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *is the one-point template.*

### 3.4.2. Eigenspaces

If $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ is definite, let $\{\mathbf{a}^{\mathrm{j_1}}, \ldots, \mathbf{a}^{\mathrm{j_k}}\}$ be a maximal set of non-equivalent fundamental eigenimages of $\mathbf{t}$. The space $< \mathbf{a}^{\mathrm{j_1}}, \ldots, \mathbf{a}^{\mathrm{j_k}} >$ generated by these eigenimages is called the *eigenspace of* $\mathbf{t}$.

**Theorem 3.54.** *Let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *be given. If the eigenproblem for* $\mathbf{t}$ *is finitely soluble then every finite eigenimage has the same unique corresponding finite eigenvalue* $\lambda$. *The template* $\mathbf{t} \boxtimes -\lambda$ *is definite, and all finite eigenimages of* $\mathbf{t}$ *lie in the eigenspace of* $\mathbf{t} \boxtimes -\lambda$. *The non-equivalent fundamental eigenimages which generate this space have the property that no one of them is linearly dependent on (any subset of) the others.*

The unique scalar in Theorem 3.54, when it exists, is called the *principal eigenvalue of* $\mathbf{t}$.

We call a bounded l-group $\mathbf{F}$ *radicable* if for each $a \in \mathbf{F}$ and integer $k \geq 1$, there exists a unique $f \in \mathbf{F}$ such that $f^k = a$.

Some examples of radicable bounded l-groups are $\mathbf{R}_{\pm\infty}$, $\mathbf{Q}_{\pm\infty}$, and $\mathbf{R}^+_{\pm\infty}$. However, $\mathbf{Z}_{\pm\infty}$ is not radicable. Choosing $a = 12$ and $k = 5$, solving for f in the equation

$$f^5 = 12$$

is just solving for f in (using regular arithmetic)

$$5f = 12$$

which, of course, has no integral solution.

Let $\mathbf{F}$ be a radicable bounded l-group, and $\mathbf{t} \in (\mathbf{F}^{\mathrm{X}}_{\pm\infty})^{\mathrm{X}}$. Let $\sigma = y_0, y_1, ..., y_m$ be a circuit in $\Delta(\mathbf{t})$. We define the *length* of $\sigma$ to be m. For each circuit $\sigma$ in $\Delta(\mathbf{t})$, of length $l(\sigma)$ and having circuit product $p(\sigma)$, we define a *circuit mean* $\mu(\sigma) \in \mathbf{F}$ by

$$[\mu(\sigma)]^{l(\sigma)} = p(\sigma).$$

We also define

$$\lambda(\mathbf{t}) = \bigvee \{ \mu(\sigma) : \sigma \text{ is a simple circuit in } \Delta(\mathbf{t}) \}.$$

For the template and associated graph $\Delta(\mathbf{t})$ in Figure 8, we have the following computations.

| Simple Circuit $\sigma$ | $p(\sigma)$ | $l(\sigma)$ | $\mu(\sigma)$ |
|---|---|---|---|
| (1,1) | 4 | 1 | 4 |
| (2,2) | -1 | 1 | -1 |
| (3,3) | 7 | 1 | 7 |
| (1,2,1) | 5 | 2 | 5/2 |
| (2,3,2) | $-\infty$ | 2 | $-\infty$ |
| (3,1,3) | $-\infty$ | 2 | $-\infty$ |
| (1,2,3,1) | 1 | 3 | 1/3 |
| (3,2,1,3) | $-\infty$ | 3 | $-\infty$ |

**Figure 8.** Computation of the Circuit Mean $\mu(\sigma)$.

In this example, $\lambda(\mathbf{t}) = 7$.

### 3.4.3. Solutions to the Eigenproblem

We now present the relation between the parameter $\lambda(\mathbf{t})$ and the principal eigenvalue for $\mathbf{t}$.

**Theorem 3.55.** *Let $\mathbf{F}_{\pm\infty}$ be a radicable bounded l-group and let $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ be given. If the eigenproblem for $\mathbf{t}$ is finitely soluble then $\lambda(\mathbf{t})$ is finite, and, in this case, $\lambda(\mathbf{t})$ is the only possible value for the eigenvalue in any finite solution to the eigenproblem for $\mathbf{t}$. That is, $\lambda(\mathbf{t})$ is the principal eigenvalue of $\mathbf{t}$.*

**Theorem 3.56.** *Let $\mathbf{F}_{\pm\infty}$ be a radicable sub-bounded l-group of $\mathbf{R}_{\pm\infty}$ and let $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ be given. Then the eigenproblem for $\mathbf{t}$ is finitely soluble if and only if $\lambda(\mathbf{t})$ is finite and the template $\mathrm{B}(\mathcal{A})$ is doubly $\mathbf{F}$-astic, where $\mathcal{A}$ =*

*$\{[\Gamma(\mathbf{t} \boxtimes -\lambda(\mathbf{t}))]'_{\mathbf{x}_{j_1}}, [\Gamma(\mathbf{t} \boxtimes -\lambda(\mathbf{t}))]'_{\mathbf{x}_{j_2}}, \ldots, [\Gamma(\mathbf{t} \boxtimes -\lambda(\mathbf{t}))]'_{\mathbf{x}_{j_k}}\}$ is a maximal set of non-equivalent fundamental eigenimages for the definite template $\mathbf{t} \boxtimes -\lambda(\mathbf{t})$.*

The Computational Task. If $|\mathbf{X}|$ is large, and $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$, then to directly evaluate the circuit product for all simple circuits in $\mathbf{t}$ is very time consuming. We now state a theorem which makes the task more manageable for the case where the bounded l-group is $\mathbf{R}_{\pm\infty}$.

**Theorem 3.57.** *Let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *be given. If the eigenproblem for* $\mathbf{t}$ *is finitely soluble, then* $\lambda(\mathbf{t})$ *is the optimal value of* $\lambda$ *in the following linear programming problem in the* n+1 *real variables* $\lambda, x_1, \ldots, x_n$:

$$\textbf{Minimize } \lambda \qquad \textbf{Subject to } \lambda + x_i - x_j \geq \mathbf{t}_{x_i}(\mathbf{x}_j)$$

*where the inequality constraint is taken over all pairs* i,j *for which* $\mathbf{t}_{x_i}(\mathbf{x}_j)$ *is finite.*

In Theorem 3.54, we noted the linear independence of the fundamental eigenimages which generate an eigenspace. We are able now to prove a stronger result which has applications to $\mathbf{R}_{\pm\infty}$ and $\mathbf{R}_{\pm\infty}^{+}$.

**Theorem 3.58** *Let* $\mathbf{F}_{\pm\infty}$ *be a radicable bounded l-group other than* $\mathbf{F}_3$, *and let* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *have a finitely soluble eigenproblem. Then the fundamental eigenimages of* $-\lambda(\mathbf{t}) \boxtimes \mathbf{t}$ *corresponding to a maximal set of non-equivalent eigennodes in* $\Delta[-\lambda(\mathbf{t}) \boxtimes \mathbf{t}]$ *are SLI.*

We now present a result relating $\lambda(\mathbf{t})$ and Inv.

**Theorem 3.59.** *Let* $\mathbf{F}_{\pm\infty}$ *be a bounded l-group and* $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$ *be such that* $\lambda(\mathbf{t}) \leq \phi$. *Then*

$$\mathrm{Inv}(\mathbf{I} \vee \mathbf{t}) = \mathbf{I} \vee \mathbf{t} \vee \mathbf{t}^2 \vee \cdots \vee \mathbf{t}^{\mathrm{K}}$$

*for arbitrary large* K. *Here,* I *denotes the identity template of* $(\mathbf{F}_{\pm\infty}^{\mathrm{X}})^{\mathrm{X}}$.

# CHAPTER 4
# GENERALIZATION OF MATHEMATICAL MORPHOLOGY

Up until the mid 1960's, the theoretical tools of quantitative microscopy as applied to image analysis were not based on any cohesive mathematical foundation. It was G. Matheron and J. Serra at the École des Mines de Paris who first pioneered the theory of mathematical morphology as a first attempt to unify the underlying mathematical concepts being used for image analysis in microbiology, petrography, and metallography [16,53,54]. Initially its main use was to describe boolean image processing in the plane, but Sternberg [55] extended the concepts in mathematical morphology to include gray valued images via the cumbersome notion of an *umbra*. While others including Serra [56,57] also extended morphology to gray valued images in different manners, Sternberg's definitions have been used more regularly, and, in fact, are used by Serra in his book [16].

The basis on which morphological theory lies are the two classical operations of Minkowski addition and Minkowski subtraction from integral geometry [13,14]. For any two sets $A \subset \mathbf{R}^n$ and $B \subset \mathbf{R}^n$, Minkowski addition and subtraction are defined as

$$A \times B = \bigcup_{b \in B} A_b \quad \text{and} \quad A \;/\; B = \bigcap_{b \in B'} A_b,$$

respectively, where $A_b = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A\}$ and $B' = \{-\mathbf{b} : \mathbf{b} \in B\}$. We have used the original notation as found in Hadwiger's book [14]. It can be shown that

$$A \;/\; B = (A^c \times B')^c,$$

where $A^c$ denotes the complement of A in $\mathbf{R}^n$. From these definitions are constructed the two morphological operations of dilation and erosion. As used by Serra and Maragos [16,21],

the *dilation* of a set $A \subset \mathbf{R}^n$ by a structuring element $B \subset \mathbf{R}^n$ is denoted by $A \boxplus B'$ and defined by

$$A \boxplus B' = \bigcup_{b \in B'} A_b$$

while *erosion* of A by B is

$$A \boxminus B = \bigcap_{b \in B'} A_b = (A^c \boxplus B)^c.$$

We remark that the actual symbols used in Serra's and Maragos' papers for the dilation and erosion are $\oplus$ and $\ominus$. To avoid confusion with the image algebra operation $\oplus$, we have replaced $\oplus$ and $\ominus$ with $\boxplus$ and $\boxminus$ respectively.

To avoid anomalies without practical interest, the structuring element B is assumed to include the origin $\mathbf{0} \in \mathbf{R}^n$, and both A and B are assumed to be compact. Unfortunately, the definitions for dilation and erosion defined by Serra are not the same as the Minkowski operations. In addition, while Maragos uses the same definitions as Serra for dilation and erosion, Maragos [21] uses the identical symbols $\boxplus$ and $\boxminus$ when defining Minkowski addition and subtraction. To add to the confusion, Sternberg defines an erosion and dilation using the same symbols $\boxplus$ and $\boxminus$ which are exactly the Minkowski operations [58]. The following table lists the three definitions. In all cases, $A_b = \{\mathbf{a} + \mathbf{b} : \mathbf{a} \in A\}$, $B' = \{-\mathbf{b} : \mathbf{b} \in B\}$, and $A^c$ denotes the complement of A in $\mathbf{R}^n$.

**Table 2.**

| | addition | subtraction |
|---|---|---|
| Minkowski | $A \times B = \bigcup_{b \in B} A_b$ | $A \,/\, B = \bigcap_{b \in B'} A_b = (A^c \boxplus B')^c$ |
| Serra Maragos | dilation of A by B<br>$A \boxplus B' = \bigcup_{b \in B'} A_b$ | erosion of A by B<br>$A \boxminus B' = \bigcap_{b \in B'} A_b = (A^c \boxplus B)^c$ |
| Sternberg | dilation of A by B<br>$A \boxplus B = \bigcup_{b \in B} A_b$ | erosion of A by B<br>$A \boxminus B = \bigcap_{b \in B'} A_b = (A^c \boxplus B')^c$ |

Thus we see that while Sternberg's dilation of A by B is exactly Minkowski's addition of A and B, Serra's dilation of A by B is Minkowski's addition of A and B′. Although both definitions of erosion of A by B are equivalent to Minkowski's subtraction of A and B, Serra uses the symbol B′ while Sternberg uses simply B. For the remainder of this chapter we will use Sternberg's definitions of dilation and erosion.

All morphological transformations are combinations of dilations and erosions, such as the *opening* of A by B, denoted by A o B,

$$A \circ B = (A \boxminus B) \boxplus B$$

and the *closing* of A by B, denoted by $A \bullet B$,

$$A \bullet B = (A \boxplus B) \boxminus B.$$

However, a more general image transform in mathematical morphology is the *Hit or Miss transform* [54,53]. Since an erosion and hence a dilation is a special case of the Hit or Miss

transform, this transform is often viewed as the universal morphological transformation upon which the theory of mathematical morphology is based. Let $B = (D,E)$ be a pair of structuring elements. Then the Hit or Miss transform of the set A is given by the expression

$$A \circledast B = \{ \mathbf{a} : D_{\mathbf{a}} \subset A,\ E_{\mathbf{a}} \subset A^c \}.$$

For practical applications it is assumed that $D \cap E = \emptyset$. The erosion of A by D is obtained by simply letting $E = \emptyset$, in which case we have $A \circledast B = A \boxminus D$.

While there have been several extensions of the boolean dilation to the gray level case, Sternberg's formulaes for computing the gray value erosion and dilation are the most straightforward, although the underlying theory introduces the somewhat extraneous concept of an *umbra*. Let f: $\mathbf{R}^n \rightarrow \mathbf{R}$ be a function. Then the *umbra* of f, denoted by $\mathcal{U}(f)$, is the set $\mathcal{U}(f) \subset \mathbf{R}^{n+1}$ defined by

$$\mathcal{U}(f) = \{\mathbf{p} = (\mathbf{x}, z) \in \mathbf{R}^{n+1} : z \leq f(\mathbf{x}) \}.$$

Again, the notion of an unbounded set is exhibited in this definition, for in general the value z can approach $-\infty$. Since $\mathcal{U}(f) \subset \mathbf{R}^k$, the dilation of two functions f and g is defined through the dilation of their umbras,

$$\mathcal{U}(f \boxplus g) \;=\; \mathcal{U}(f) \boxplus \mathcal{U}(g),$$

and similarly the erosion of f by g,

$$\mathcal{U}(f \boxminus g) \;=\; \mathcal{U}(f) \boxminus \mathcal{U}(g).$$

Any function d: $\mathbf{R}^n \rightarrow \mathbf{R}$ has the property that $d(\mathbf{x}) = \max\{z \in \mathbf{R} : (\mathbf{x}, z) \in \mathcal{U}(d)\}$, and thus the set $\mathcal{U}(f \boxplus g)$ well-defines the function $f \boxplus g$. However, when actually calculating the new functions $d = f \boxplus g$ and $e = f \boxminus g$, Sternberg gives the following formulae for the two-dimensional dilation and erosion, respectively:

$$d(x,y) \;=\; \max_{i,j}\,[\,f(x-i,y-j)+g(i,j)\,] \tag{4-1}$$

$$e(x,y) \;=\; \min_{i,j}\,[\,f(x-i,y-j)-g(-i,-j)\,] \tag{4-2}$$

The function f represents the image, and g represents the structuring element. Both f and g are assumed to have finite support, with values of $-\infty$ outside. Also, in general the support of g is much smaller than the coordinate set $\mathbf{X}$, and $g(\mathbf{0}) \neq -\infty$. So in practice, the notion of an umbra need not be introduced at all.

Note that when applying these transforms to real data, we cannot simply substitute an image $\mathbf{a}$ for the set A, as the expression $A^c$ becomes meaningless to a computer. What is actually assumed is that A corresponds to the black pixels in a boolean image $\mathbf{a}$, that is, given $A \subset \mathbf{R}^n$, a coordinate set $\mathbf{X} \subset \mathbf{R}^n$ is chosen and a two-valued image $\mathbf{a}$ on $\mathbf{X}$ is found, where 1 and 0 represent the two values:

$$\mathbf{a}(\mathbf{x}) \;=\; \begin{cases} 1 & \text{if } \mathbf{x} \in A \subset \mathbf{X} \\ 0 & \text{otherwise} \end{cases} .$$

For the two-dimensional gray value case, Sternberg's formulas (4-1) and (4-2) are easily written in computer code, and this is, in fact, close to the image algebra definition for dilation. In short, when implementing a problem which is posed in morphological terms, the solution must be reposed in a setting which more closely represents the computing environment. On the other hand, it has been established that the image algebra comes very close to ideally modeling a large number of important image processing problems, such as mapping of transforms to sequential and parallel architectures [44] and this dissertation, and expressing sequential algorithms in a parallel manner [59].

The next part of this chapter is devoted to establishing an isomorphism between the morphological algebra and the image algebra. We will show that performing a dilation is equivalent to calculating

$$\mathbf{a} \boxtimes \mathbf{t}$$

for the appropriate **a** and **t**, and performing an erosion is equivalent to calculating

$$\mathbf{a} \boxtimes \mathbf{t}^*$$

for appropriate **a** and **t**.

Let A,B be finite subsets of $\mathbf{Z}^n$, where B is a structuring element. Let $\mathbf{X} = \mathbf{Z}^n$ or choose $\mathbf{X} \subset \mathbf{Z}^n$ to be a finite set such that $A \boxplus B \subset \mathbf{X}$. Let $\mathbf{F}_4$ denote the value set $\{-\infty, 0\ 1, +\infty\}$. Define $\xi\colon 2^{\mathbf{Z}^n} \to \mathbf{F}_4^{\mathbf{X}}$ by $\xi(A) = \mathbf{a}$ where

$$\mathbf{a}(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in A \\ 0 & \text{otherwise} \end{cases}.$$

Let $\mathcal{B} = \{ B \subset \mathbf{Z}^n : |B| < \infty \text{ and } \mathbf{0} \in B \}$, and let $\mathcal{I}$ be the set of all $\mathbf{F}_4$ valued invariant templates from $\mathbf{X}$ to $\mathbf{X}$ such that $\mathbf{y} \in \mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y})$. Define $\eta\colon \mathcal{B} \to \mathcal{I}$ by $\eta(B) = \mathbf{t}$ where

$$\mathbf{t}_\mathbf{y}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x} \in B'_\mathbf{y} \\ -\infty & \text{otherwise} \end{cases}.$$

**Lemma 4.1.** *Let $\xi$, $\eta$ be as above. Let $A \subset \mathbf{Z}^n$, and $B \in \mathcal{B}$ a structuring element. Then*

$$\xi(A \boxplus B) = \xi(A) \boxtimes \eta(B).$$

**Proof:** Choose $\mathbf{X}$ large enough such that $A \boxplus B \subset \mathbf{X}$. Let $D = A \boxplus B$ and $\mathbf{f} = \xi(A) \boxtimes \eta(B)$. We must show that $\mathbf{y} \in D$ if and only if $\mathbf{f}(\mathbf{y}) = 1$. To this end, we note that

$$\mathbf{y} \in A \boxplus B \iff \mathbf{y} \in A_\mathbf{b} \text{ for some } \mathbf{b} \in B \iff \mathbf{y} = \mathbf{x} + \mathbf{b} \text{ for } \mathbf{x} \in A,\ \mathbf{b} \in B$$

$$\iff \mathbf{x} = (-\mathbf{b}) + \mathbf{y},\ -\mathbf{b} \in B',\ \mathbf{x} \in A \iff \mathbf{x} \in A \text{ and } \mathbf{x} = (-\mathbf{b}) + \mathbf{y} \in B'_\mathbf{y}$$

$$\iff \mathbf{a}(\mathbf{x}) = 1 \text{ and } \mathbf{t}_\mathbf{y}(\mathbf{x}) = 0 \iff \bigvee_{\mathbf{z} \in \mathbf{X}} \mathbf{a}(\mathbf{z}) + \mathbf{t}_\mathbf{y}(\mathbf{z}) = \mathbf{f}(\mathbf{y}) = 1.$$

Q.E.D.

We call **a** the *image corresponding to* A, and **t** the *template corresponding to the structuring element* B.

The next lemma shows the correspondence between the $\boxtimes$ operation and erosion.

**Lemma 4.2.** *Let $\xi$, $\eta$ be as above. Let $A \subset \mathbf{Z}^n$, and $B \subset \mathbf{Z}^n$ a structuring element. Then*

$$\xi(A \boxminus B) = \xi(A) \boxtimes [\eta(B)]^*.$$

**Proof:** Let $D = A \boxminus B$ and let $\mathbf{c} = \xi(A) \boxtimes [\eta(B)]^*$. We must show that $\mathbf{y} \in D$ if and only if $\mathbf{c}(\mathbf{y}) = 1$.

$$\mathbf{y} \in D \iff \mathbf{y} \in A_{\mathbf{p}}\ \forall\, \mathbf{p} \in B' \iff \mathbf{y} = \mathbf{x}_{\mathbf{p}} + \mathbf{p}\ \forall\, \mathbf{p} \in B',$$

where the choice of $\mathbf{x}_{\mathbf{p}} \in A$ depends on $\mathbf{p}$. Let $\mathbf{a} = \xi(A)$ and $\mathbf{t} = \eta(B)$. Then

$\mathbf{c} = \mathbf{a} \boxtimes \mathbf{t}^*$ and

$$\mathbf{c}(\mathbf{y}) = \bigwedge_{\mathbf{x} \in X} \mathbf{a}(\mathbf{x}) +' \mathbf{t}_{\mathbf{y}}^*(\mathbf{x}) = \bigwedge_{\mathbf{x} \in S_{+\infty}(\mathbf{t}_{\mathbf{y}}^*)} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}^*(\mathbf{x}).$$

We have

$$\mathbf{t}_{\mathbf{y}}^*(\mathbf{x}) = [\mathbf{t}_{\mathbf{x}}(\mathbf{y})]^* = \begin{cases} 0 & \text{if } \mathbf{y} \in B'_{\mathbf{x}} \\ +\infty & \text{otherwise} \end{cases}.$$

We claim that $S_{+\infty}(\mathbf{t}_{\mathbf{y}}^*) = B_{\mathbf{y}}$. To show this, note that

$$\mathbf{x} \in S_{+\infty}(\mathbf{t}_{\mathbf{y}}^*) \iff \mathbf{t}_{\mathbf{y}}^*(\mathbf{x}) = 0 = \mathbf{t}_{\mathbf{x}}(\mathbf{y}) \iff \mathbf{y} \in B'_{\mathbf{x}} \iff$$

$$\mathbf{y} = \mathbf{p} + \mathbf{x} \text{ for some } \mathbf{p} \in B' \iff \mathbf{x} = \mathbf{b} + \mathbf{y} \text{ for some } \mathbf{b} \in B \iff \mathbf{x} \in B_{\mathbf{y}}.$$

Thus,

$$\mathbf{y} \in D \iff \mathbf{y} = \mathbf{x}_{\mathbf{p}} + \mathbf{p}\ \forall\, \mathbf{p} \in B' \iff \mathbf{x}_{\mathbf{b}} = \mathbf{b} + \mathbf{y}\ \forall\, \mathbf{b} \in B, \text{ for some } \mathbf{x}_{\mathbf{b}} \in A$$

$$\iff \mathbf{b} + \mathbf{y} = \mathbf{x} \in A\ \forall\, \mathbf{b} \in B \iff B_{\mathbf{y}} = S_{+\infty}(\mathbf{t}_{\mathbf{y}}^*) \subset A \text{ (by definition of } \boxminus) \iff$$

$$\mathbf{a}(\mathbf{x}) = 1\ \forall\, \mathbf{x} \in B_{\mathbf{y}} \subset A \text{ and } \mathbf{t}_{\mathbf{y}}^*(\mathbf{x}) = 0\ \forall\, \mathbf{x} \in B_{\mathbf{y}} = S_{+\infty}(\mathbf{t}_{\mathbf{y}}^*) \iff$$

$$\bigwedge_{\mathbf{x} \in S_{+\infty}(\mathbf{t}_{\mathbf{y}}^*)} \mathbf{a}(\mathbf{x}) + \mathbf{t}_{\mathbf{y}}^*(\mathbf{x}) = 1 = \mathbf{c}(\mathbf{y}).$$

Q.E.D.

Lemmas 4.1 and 4.2 include not only boolean but gray level dilation and erosion. However, we now show explicitly that Sternberg's formulas (4-1) and (4-2) hold in the two dimensional case. Let f,g $:\mathbf{Z}^2 \rightarrow \mathbf{R}_{-\infty}$ be two real extended real valued functions with finite support, where f represents the image and g the structuring element. Choose $\mathbf{X}$ to be either $\mathbf{Z}^2$ or a finite subset containing the support of $f \boxplus g$. Then $\xi : \mathbf{R}_{-\infty}^{\mathbf{Z}^2} \rightarrow \mathbf{R}_{-\infty}^{\mathbf{X}}$ is the identity function restricted to $\mathbf{X}$, that is, $\mathbf{a}(\mathbf{x}) = f(\mathbf{x})$, where $\mathbf{a} = \xi(f)$. Let B denote the support of g, $B = \{ \mathbf{x} \in \mathbf{X} : g(\mathbf{x}) \neq -\infty \}$. Let $\mathcal{T}$ denote the set of all $\mathbf{R}_{\pm\infty}$ valued templates from $\mathbf{X}$ to $\mathbf{X}$ such that $\mathbf{y} \in \mathcal{S}_{-\infty}(\mathbf{t_y})$ for all $\mathbf{y} \in \mathbf{X}$. Define $\eta : \mathbf{R}_{-\infty}^{\mathbf{Z}^2} \rightarrow \mathcal{T}$ by $\eta(g) = \mathbf{t}$ where

$$\mathbf{t_y}(\mathbf{x}) = \begin{cases} g(\mathbf{y}-\mathbf{x}) & \text{if } \mathbf{x} \in B'_{\mathbf{y}} \\ -\infty & \text{otherwise} \end{cases}.$$

Note that if $\mathbf{x} \in B'_{\mathbf{y}}$, then $\mathbf{x} = \mathbf{p} + \mathbf{y}$ for some $\mathbf{p} \in B'$, which implies that $g(-\mathbf{p}) = g(\mathbf{y}-\mathbf{x})$ is well-defined. Also, $\mathcal{S}_{-\infty}(\mathbf{t_y}) = B'_{\mathbf{y}}$. The formal relation between the gray-scale morphological operations and $\boxtimes$ and $\boxslash$ are shown in the next two theorems.

**Theorem 4.3.** *Let* f,g, B, *and* $\mathbf{X}$ *be as above. Then*

$$\xi(f \boxplus g) \;=\; \xi(f) \boxplus \eta(g).$$

**Proof:**

At location $(x,y) \in \mathbf{X} \subset \mathbf{Z}^2$,

$$(\mathbf{a} \boxtimes \mathbf{t})(x,y) \;=\; \bigvee_{\mathbf{z} \in \mathbf{X}} \mathbf{a}(\mathbf{z}) + \mathbf{t}_{(x,y)}(\mathbf{z}) \;=\; \bigvee_{(i,j) \in \mathcal{S}_{-\infty}(\mathbf{t}_{(x,y)})} \mathbf{a}(i,j) + \mathbf{t}_{(x,y)}(i,j),$$

while at (x,y), $f \boxplus g = d$ has value

$$d(x,y) \;=\; \max_{(i,j) \in B} [f(x-i, y-j) + g(i,j)] \;=\; \max_{(x-i,y-j) \in B} [f(i,j) + g(x-i,y-j)].$$

Given $(x-i,y-j) = (-p_1,-p_2) \in B$, we have $(i,j) = (p_1, p_2) + (x,y) \in B'_{(x,y)}$, and hence,

$$f(i,j) + g(x-i,y-j) = \mathbf{a}(i,j) + \mathbf{t}_{(x,y)}(i,j) \;\; \forall\, (i,j) \in B'_{(x,y)}.$$

Therefore,

$$d(x,y) \;=\; \max_{(x-i,y-j)\in B}[f(i,j) + g(x-i,y-j)] \;=\; \max_{(i,j)\in B'_{(x,y)}}[f(i,j) + g(x-i,y-j)]$$

$$= \bigvee_{(i,j)\in S_{-\infty}(\mathbf{t}_{(x,y)})} \mathbf{a}(i,j) + \mathbf{t}_{(x,y)}(i,j) \;=\; (\mathbf{a} \boxtimes \mathbf{t})(x,y).$$

Q.E.D.

If the template $\mathbf{t}$ corresponding to the structuring element g has form

$$\mathbf{t}_{\mathbf{y}}(\mathbf{x}) = \begin{cases} g(\mathbf{y}-\mathbf{x}) & \text{if } \mathbf{x} \in B'_{\mathbf{y}} \\ -\infty & \text{otherwise} \end{cases},$$

then the template $\mathbf{t}^*$ has form

$$\mathbf{t}^*_{\mathbf{y}}(\mathbf{x}) = \begin{cases} -g(\mathbf{x}-\mathbf{y}) & \text{if } \mathbf{y} \in B'_{\mathbf{x}} \\ +\infty & \text{otherwise} \end{cases}.$$

Since $\mathbf{t}^*_{\mathbf{y}}(\mathbf{x}) \in \mathbf{R}$ if and only if $\mathbf{x}-\mathbf{y} \in B$ if and only if $\mathbf{x} \in B_{\mathbf{y}}$, $S_{+\infty}(\mathbf{t}^*_{\mathbf{y}}) = B_{\mathbf{y}}$.

**Theorem 4.4.** *Let* f,g, B, *and* $\mathbf{X}$ *be as in Theorem* 4.3. *Then*

$$\xi(f \boxminus g) \;=\; \xi(f) \boxminus [\eta(g)]^*.$$

**Proof:**

At location $(x,y) \in \mathbf{X} \subset \mathbf{Z}^2$,

$$(\mathbf{a} \boxtimes \mathbf{t}^*)(x,y) \;=\; \bigwedge_{\mathbf{z}\in\mathbf{X}} \mathbf{a}(\mathbf{z}) +' \mathbf{t}^*_{(x,y)}(\mathbf{z}) \;=\; \bigwedge_{(i,j)\in S_{+\infty}(\mathbf{t}^*_{(x,y)})} \mathbf{a}(i,j) + \mathbf{t}^*_{(x,y)}(i,j),$$

while at (x,y), $f \boxminus g = e$ has value

$$e(x,y) \;=\; \min_{(i,j)\in B'}[f(x-i,y-j) - g(-i,-j)] \;=\; \min_{(-x+i,-y+j)\in B}[f(i,j) - g(-x+i,-y+j)].$$

Given $(-x+i,-y+j) = (b_1,b_2) \in B$, we have $(i,j) = (b_1,b_2) + (x,y) \in B$, and, hence,

$$f(i,j) - g(-x+i,-y+j) = \mathbf{a}(i,j) + \mathbf{t}^*_{(x,y)}(i,j) \;\; \forall\, (i,j) \in B_{(x,y)}.$$

Therefore,

$$e(x,y) \;=\; \min_{(-x+i,-y+j)\in B}[f(i,j) - g(-x+i,-y+j)] \;=\; \min_{(i,j)\in B_{(x,y)}}[f(i,j) - g(-x+i,-y+j)]$$

$$= \bigwedge_{(i,j)\in S_{+\infty}(\mathbf{t}^*_{(x,y)})} \mathbf{a}(i,j) + \mathbf{t}^*_{(x,y)}(i,j) \;=\; (\mathbf{a} \boxtimes \mathbf{t}^*)(x,y).$$

Q.E.D.

It is easily ascertained that each of $\xi$ and $\eta$ are one-one and onto for each of the boolean and gray level cases. The functions $\xi$ and $\eta$ therefore preserve the morphological operations, and in fact theorems 4.1 through 4.4 show that mathematical morphology is embedded into the image algebra. We condense the results in the following two expressions.

| | | |
|---|---|---|
| $\mathbf{a} \boxed{\vee} \mathbf{t}$ | corresponds to the dilation of f by g, | $f \boxplus g$ |
| $\mathbf{a} \boxed{\wedge} \mathbf{t}^{*}$ | corresponds to the erosion of f by g, | $f \boxminus g$ |

The operation $\ovee$ can also be used to express a boolean dilation or erosion. Given A $\subset \mathbf{R}^n$, the image $\mathbf{a} \in \{0,1\}^X$ corresponding to A is defined as before, while for a structuring element $B \subset \mathbf{R}^n$, the template $\mathbf{t}$ corresponding to B is defined by

$$\mathbf{t}_y(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{x} \in B'_y \\ 0 & \text{otherwise} \end{cases}.$$

The $\mathbf{a} \ovee \mathbf{t}$ corresponds to the dilation of A by B, while $\mathbf{a} \owedge \bar{\mathbf{t}}$ corresponds to the erosion of A by B. Here, of course,

$$\bar{\mathbf{t}}_y(\mathbf{x}) = \begin{cases} 1 & \text{if } \mathbf{y} \in B'_x \\ +\infty & \text{otherwise} \end{cases}.$$

Thus, the value set $\{ -\infty, 0, 1, +\infty \}$ along with the operation $\boxed{\vee}$ can be used to express a boolean dilation, where both input image $\mathbf{a}$ and output $\mathbf{a} \boxed{\vee} \mathbf{t}$ are $\{ 0,1 \}$ valued images. Similarly, the three element blog $\mathbf{F}_3 = \{ 0,1,+\infty \}$ along with the operation $\ovee$ can be used to express a boolean dilation, where both $\mathbf{a}$ and $\mathbf{a} \ovee \mathbf{t}$ are $\{ 0,1 \}$ valued images.

In the boolean case it is a simple exercise to show that the Hit or Miss transform can be expressed as

$$A \circledast B = (A \boxminus D) \cap (A^c \boxminus E). \tag{4-3}$$

If we let $\mathbf{t}$ be the boolean template corresponding to D and $\mathbf{s}$ the boolean template corresponding to E, we obtain the equivalent image algebra expression

$$(\mathbf{a} \boxtimes \mathbf{t}^*) * (\chi_0(\mathbf{a}) \boxtimes \mathbf{s}^*), \tag{4-4}$$

or, using the bounded l-group $\mathbf{F}_3 = \{0, 1, +\infty\}$,

$$(\mathbf{a} \oslash \bar{\mathbf{t}}) * (\chi_0(\mathbf{a}) \oslash \bar{\mathbf{s}}). \tag{4-5}$$

However, there is an even simpler image algebra formulation of the Hit or Miss transform which does not employ the notions of minimum or erosion. We state this in the next lemma, using equation (4-5) as the representation of the Hit or Miss transform. Before doing so, however, we introduce the concept of a *census template*. Let $S \subset \mathbf{X} \subset \mathbf{R}^n$, where $\mathbf{X}$ is a finite array. Suppose every vector $\mathbf{s} = (s_1, \ldots, s_n) \in S$ is assigned a value of 0 or 1, and we wish to determine if the pattern of 0's and 1's within S matches some predetermined pattern we want to identify. One can uniquely identify every possible combination of 0 and 1 values within S by attaching a unique number to each combination. This is done by using an invariant template whose support is the set S and each weight $\mathbf{t}_\mathbf{y}(\mathbf{x})$, $\mathbf{x} \in S$, is a distinct power of a prime number. When applied via the operation $\oplus$ to a boolean image, each distinct pattern of 0's and 1's in S will result in a unique number. While this theory is general enough to identify patterns in n-dimensional space, it is primarily used for 2-dimensional image processing. The following is a 2-dimensional census template.

| | | |
|---|---|---|
| 16 | 8 | 4 |
| 32 | 1 | 2 |
| 64 | 128 | 256 |

**Figure 9.** An Example of a Census Template.

This template was used as part of an algorithm to determine the chain code of a boolean image [59].

**Lemma 4.5.** *Let* t *be the boolean template corresponding to* D *and* **s** *the boolean template corresponding to* E *in the Hit or Miss transform* (4-5). *Suppose* $\mathcal{S}_{+\infty}(\bar{\mathbf{t}}_{\mathbf{y}}) = \{\mathbf{x}_1, \ldots, \mathbf{x}_k\}$ *and* $\mathcal{S}_{+\infty}(\bar{\mathbf{s}}_{\mathbf{y}}) = \{\mathbf{x}_{k+1}, \ldots, \mathbf{x}_n\}$, *where the* $\mathbf{x}_i$ *are distinct,* $i = 1,\ldots,n$. *Then*

$$\chi_m(\mathbf{a} \oplus \mathbf{r}), \text{ where } m = \sum_{i=1}^{k} 2^{i-1},$$

*is equivalent to computing the Hit or Miss transform, and the* n*-point census template* $\mathbf{r} \in (\mathbf{R}^{\mathbf{X}})^{\mathbf{X}}$ *is defined by*

$$\mathbf{r}_{\mathbf{y}}(\mathbf{x}_i) = \begin{cases} 2^{i-1} & i=1,\ldots,n \\ 0 & \mathbf{x} \neq \mathbf{x}_i \text{ for any } i = 1,\ldots,n \end{cases}.$$

**Proof:** Note that $\mathbf{a} \in \{0,1\}^{\mathbf{X}}$ and $\chi_m(\mathbf{a} \oplus \mathbf{r}) \in \{0,1\}^{\mathbf{X}}$ also. Let $\mathbf{b} = (\mathbf{a} \circledA \bar{\mathbf{t}}) * (\chi_0(\mathbf{a}) \circledA \bar{\mathbf{s}})$, and let $\mathbf{c} = \chi_0(\mathbf{a})$. Then at $\mathbf{y} \in \mathbf{X}$, expression (4-5) has value

$$\mathbf{b}(\mathbf{y}) = [\bigwedge_{\mathbf{x} \in \mathcal{S}_{+\infty}(\bar{\mathbf{t}}_{\mathbf{y}})} \mathbf{a}(\mathbf{x}) * \bar{\mathbf{t}}_{\mathbf{y}}(\mathbf{x})] * [\bigwedge_{\mathbf{x} \in \mathcal{S}_{+\infty}(\bar{\mathbf{s}}_{\mathbf{y}})} \mathbf{c}(\mathbf{x}) * \bar{\mathbf{s}}_{\mathbf{y}}(\mathbf{x})]$$

$$= \left[\bigwedge_{i=1}^{k} \mathbf{a}(\mathbf{x}_i) * \bar{\mathbf{t}}_{\mathbf{y}}(\mathbf{x}_i)\right] * \left[\bigwedge_{i=k+1}^{n} \mathbf{c}(\mathbf{x}_i) * \bar{\mathbf{s}}_{\mathbf{y}}(\mathbf{x}_i)\right].$$

The pixel $\mathbf{b}(\mathbf{y})$ will have value 1 if and only if each of the expressions

$$\left[ \bigwedge_{i=1}^{k} \mathbf{a}(\mathbf{x}_i) * \bar{\mathbf{t}}_y(\mathbf{x}_i) \right] \tag{4-6}$$

$$\left[ \bigwedge_{i=k+1}^{n} \mathbf{c}(\mathbf{x}_i) * \bar{\mathbf{s}}_y(\mathbf{x}_i) \right] \tag{4-7}$$

has value 1. Note that the only other possible value that (4-6) or (4-7) can assume is 0. Now,

$$\left[ \bigwedge_{i=1}^{k} \mathbf{a}(\mathbf{x}_i) * \bar{\mathbf{t}}_y(\mathbf{x}_i) \right] = 1 \iff \mathbf{a}(\mathbf{x}) * \bar{\mathbf{t}}_y(\mathbf{x}) = 1 \ \forall \ i = 1,\ldots,k,$$

$$\iff \mathbf{a}(\mathbf{x}_i) = 1 \text{ and } \bar{\mathbf{t}}_y(\mathbf{x}_i) = 1 \ \forall \ i = 1,\ldots,k.$$

Also,

$$\left[ \bigwedge_{i=k+1}^{n} \mathbf{c}(\mathbf{x}_i) * \bar{\mathbf{s}}_y(\mathbf{x}_i) \right] = 1 \iff \mathbf{c}(\mathbf{x}) * \bar{\mathbf{s}}_y(\mathbf{x}) = 1 \ \forall \ i = k+1,\ldots,n,$$

$$\iff \mathbf{c}(\mathbf{x}_i) = 1 \text{ and } \bar{\mathbf{s}}_y(\mathbf{x}_i) = 1 \ \forall \ i = k+1,\ldots,n.$$

Thus, expression (4-5) will have value 1 if and only if $\mathbf{a}(\mathbf{x}_i) = 1$ for all $i = 1,\ldots,k$ and $\mathbf{c}(\mathbf{x}_i) = 1$ for all $i = k+1,\ldots,n$. But $\mathbf{c}(\mathbf{x}_i) = 1$ if and only if $\mathbf{a}(\mathbf{x}_i) = 0$, and this is true for all $i = k+1,\ldots,n$. Therefore (4-5) will have value 1 if and only if $\mathbf{a}(\mathbf{x}_i) = 1$ for all $i = 1,\ldots,k$ and $\mathbf{a}(\mathbf{x}_i) = 0$ for all $i = k+1,\ldots,n$. The image $\chi_m(\mathbf{a} \oplus \mathbf{r})$ will assume a non-zero value only when $\mathbf{a} \oplus \mathbf{r}$ has value m. This happens if and only if $\mathbf{a}(\mathbf{x}_i) = 1$ for all $i = 1,\ldots,k$ and $\mathbf{a}(\mathbf{x}_i) = 0$ for all $i = k+1,\ldots,n$, as

$$(\mathbf{a} \oplus \mathbf{r})(\mathbf{y}) = \sum_{x \in \mathcal{S}(\mathbf{r}_y)} \mathbf{a}(\mathbf{x}) \cdot \mathbf{r}_y(\mathbf{x}) = \sum_{i=1}^{n} \mathbf{a}(\mathbf{x}_i) \cdot \mathbf{r}_y(\mathbf{x}_i) = \sum_{i=1}^{n} \mathbf{a}(\mathbf{x}_i) \cdot 2^{i-1}$$

$$= \begin{cases} m = \sum_{i=1}^{k} 2^{i-1} & \text{if and only if } \mathbf{a}(\mathbf{x}_i) = 1 \text{ for } i = 1,\ldots,k \text{ and} \\ & \qquad \mathbf{a}(\mathbf{x}_i) = 0 \text{ for } i = k+1,\ldots,n \\ \text{an integer} \neq m & \text{otherwise} \end{cases} .$$

Here we use the fact that $\mathcal{S}(\mathbf{r}_y) = \mathcal{S}_{+\infty}(\bar{\mathbf{t}}_y) \cup \mathcal{S}_{+\infty}(\bar{\mathbf{s}}_y)$, and also that

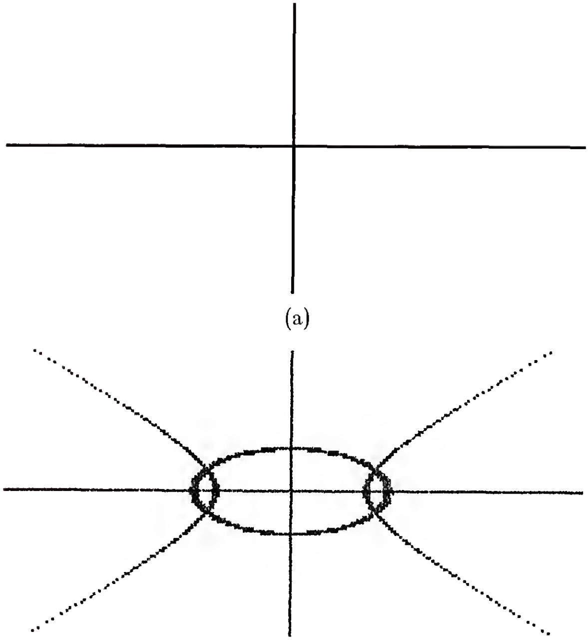
$\mathcal{S}_{+\infty}(\bar{\mathbf{t}}_{\mathbf{y}}) \cap \mathcal{S}_{+\infty}(\bar{\mathbf{s}}_{\mathbf{y}}) = \emptyset \; \forall \; \mathbf{y} \in \mathbf{X}$ (as $D \cap E = \emptyset$). Therefore we see that (4-5) will have value 1 at location $\mathbf{y}$ if and only if $\chi_m(\mathbf{a} \oplus \mathbf{r})$ has value 1 at location $\mathbf{y}$.

Q.E.D.

We have shown that the subalgebra $\mathcal{A} = (\mathbf{R}^{\mathbf{X}}, \mathcal{I}, \vee, \boxed{\vee}, \wedge, \boxed{\wedge})$ of the full image algebra is isomorphic to the morphological algebra as described by Serra and Sternberg. Since invariant templates with the target pixel included in their support (the set $\mathcal{I}$) are a special type of templates, it is clear that $(\mathbf{R}^{\mathbf{X}}, (\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{X}}, \vee, \boxed{\vee}, \wedge, \boxed{\wedge})$ is a much larger algebra than the morphological algebra. Templates generalize the concept of a structuring element. Templates can vary in size, shape, and weights from point to point and they are able to express a more general mapping, taking an image with possibly values of $+\infty$ in m-dimensional space to an image in n-dimensional space if we replace $(\mathbf{R}^{\mathbf{X}}_{-\infty})^{\mathbf{X}}$ by $(\mathbf{R}^{\mathbf{X}}_{\pm\infty})^{\mathbf{Y}}$, where $\mathbf{X} \subset \mathbf{R}^m$ and $\mathbf{Y} \subset \mathbf{R}^n$. Thus, an expression of form $\mathbf{a} \boxed{\vee} \mathbf{t}$ can represent a far more complex process than a simple dilation. For example, let $\mathbf{a}$ denote the input image shown in Figure 10(a) and define $\mathbf{t}$ by

$$\mathbf{t}_{(x,y)}(i,j) = \begin{cases} 0 & \text{if } (x,y) = (i,j) \text{ or} \\ & \quad \text{if } (x,y) \in S_1 \cup S_2 \text{ and } (0,0) = (i,j) \\ -\infty & \text{otherwise} \end{cases}$$

where $S_1 = \{ (x,y) : 0.9 < \frac{x^2}{p^2} + \frac{y^2}{q^2} < 1.1 \}$ and $S_2 = \{ (x,y) : 0.9 < \frac{x^2}{d^2} - \frac{y^2}{q^2} < 1.1 \}$, c = 30, q=15, $p^2 = q^2 + c^2$, $d^2 = c^2 - q^2$. In this case $\mathbf{a} \boxed{\vee} \mathbf{t}$ is obviously not a simple dilation. It is not at all clear if this transformation can be expressed in terms of dilations and erosions, starting with the input image $\mathbf{a}$, and if this is indeed possible, if the resulting expression would be transparent enough to justify the effort. The input and output images are shown in the next two figures.



(a)

(b)

**Figure 10.** Example of a Non-morphological Transformation.
(a) Input Image $\mathbf{a}$; (b) Image $\mathbf{a} \boxtimes \mathbf{t}$.

As a last remark, we state that a structuring element corresponds to a square matrix that is block toeplitz with toeplitz blocks that has finite elements on the diagonal. A decomposition technique for a class of invariant templates called rectangular templates is discussed in Chapter 5, and presents the method in matrix as well as image algebra notation.

# CHAPTER 5
# TRANSFORM DECOMPOSITION

## 5.1. New Matrix Decomposition Results

In this section, we state matrix results which do not appear in the book *Minimax Algebra* and which are new results. We will have particular use for most of the material presented here, as it will be used to give necessary and sufficient conditions for local decomposition of lattice transforms. The isomorphisms are used to map matrix algebra techniques to the image algebra.

As mentioned in the introduction, the use of parallel processors in computing image processing transforms is on the increase. Most transforms are not able to be applied directly to a parallel architecture. Instead, a particular transform must usually be *mapped* to a specific architecture, that is, the limitations of the machinery upon which the transform is to be implemented must be represented in the mathematical expression of the transform. For example, in the case of the neighborhood array processors, this involves *decomposing* the transform into a sequence of factors where each factor is directly implementable on the architecture. To give the general idea of this approach, we represent the set of processors by a rectangular array $\mathbf{X}$, $\mathbf{X} = \{(i,j) : 0 \leq i \leq m,\ 0 \leq j \leq n\}$, and if the neighborhood has a von Neumann or Moore configuration, then communication links between each processor and the neighboring processors to which it is connected (the *local network*) are represented by one of the diagrams in Figure 1.

Here, the processor $\mathbf{x}$ can distribute and receive information from its four (or eight) nearest neighbors. A transform $\mathbf{t}$ is represented by a template $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}$. If we

can write $\mathbf{t}$ as a product of templates,

$$\mathbf{t} = \mathbf{t}^1 \boxtimes \mathbf{t}^2 \boxtimes \cdots \boxtimes \mathbf{t}^k$$

where each $\mathbf{t}^i$ has its support $\mathcal{S}_{-\infty}(\mathbf{t}^i_{\mathbf{y}})$ as a subset of the local network configuration (in this example, the von Neumann or Moore configuration), for all $\mathbf{y} \in \mathbf{X}$, then we say that $\mathbf{t}$ has a local decomposition with respect to the network. Thus, the important consideration in determining decompositions is the underlying network of communication between processors. The network can be modeled by a graph or digraph, with nodes as the processors and edges or directed edges as the communication links between processors. The results on decomposition that follow will enable us to determine necessary and sufficient conditions on the graph of the network to guarantee the existence of a local transform decomposition under the operation of $\boxtimes$ or $\circledvee$.

In this chapter, we assume that $\mathbf{F}_{\pm\infty}$ is a sub-bounded l-group of $\mathbf{R}_{\pm\infty}$ or $\mathbf{R}^+_{\pm\infty}$, where $\mathbf{F}$ is the group of the bounded l-group $\mathbf{F}_{\pm\infty}$, and $-\infty$ and $+\infty$ have their usual meaning in context of the respective bounded l-groups. Some sub-bounded l-groups are $\mathbf{R}_{\pm\infty}$, $\mathbf{Q}_{\pm\infty}$, $\mathbf{Z}_{\pm\infty}$, and $\mathbf{Q}^+_{\pm\infty}$. Further, we assume that any matrix described in these sections, unless noted otherwise, will assume no values of $+\infty$.

### 5.1.1. Preliminaries

We first give the results in matrix notation in section 5.1, and then use the isomorphism to map the results to image algebra notation in section 5.2.

Properties of the transpose and conjugate. Recall the notation $\mathbf{t}'$ which denotes the transpose of the matrix $\mathbf{t}$, and that $(\mathbf{t}')' = \mathbf{t}$.

**Theorem 5.1.** *Let* $\mathbf{s} \in \mathcal{M}_{mp}$, $\mathbf{t} \in \mathcal{M}_{pn}$ *be given. Then* $(\mathbf{s} \times \mathbf{t})' = \mathbf{t}' \times \mathbf{s}'$, *and* $(\mathbf{s} \times' \mathbf{t})' = \mathbf{t}' \times' \mathbf{s}'$.

**Proof:** We give the proof for $(\mathbf{s} \times \mathbf{t})' = \mathbf{t}' \times \mathbf{s}'$. The case where $\times'$ replaces $\times$ is done in a similar way. First, note that $(\mathbf{s} \times \mathbf{t})' \in \mathcal{M}_{nm}$, and $\mathbf{t}' \times \mathbf{s}' \in \mathcal{M}_{nm}$ also. Let $\mathbf{r} = \mathbf{s} \times \mathbf{t}$, and $\mathbf{u} = \mathbf{t}' \times \mathbf{s}'$. To prove the lemma, it is equivalent to show that $\mathbf{r} = \mathbf{u}'$:

$$r_{ij} = u_{ji}.$$

We have

$$r_{ij} = \bigvee_{k=1}^{n} s_{ik} \times t_{kj}.$$

The elements of the j-th row of $\mathbf{t}'$ are $t_{1j}, t_{2j}, \ldots, t_{nj}$ and the elements of the i-th column of $\mathbf{s}'$ are $s_{i1}, s_{i2}, \ldots, s_{in}$. Thus, the element in the j-th row and i-th column of $\mathbf{u} = \mathbf{t}' \times \mathbf{s}'$ is

$$u_{ji} = \bigvee_{h=1}^{n} t_{hj} \times s_{ih}.$$

But

$$u_{ji} = \bigvee_{h=1}^{n} t_{hj} \times s_{ih} = \bigvee_{h=1}^{n} s_{ih} \times t_{hj} = r_{ij}.$$

Q.E.D.

**Theorem 5.2.** $\mathbf{a} \times' \mathbf{b} = (\mathbf{b}^* \times \mathbf{a}^*)^*$, *where* $\mathbf{a}$, $\mathbf{b}$ *take values in* $\mathbf{R}_{\pm\infty}$ *or* $\mathbf{R}_{\pm\infty}^{+}$.

**Proof:** For matrices of the appropriate sizes, we know

$$(\mathbf{f} \times \mathbf{g})^* = \mathbf{g}^* \times' \mathbf{f}^*,$$

as $\mathcal{M}_{mn}$ is conjugate to $\mathcal{M}_{mn}^*$ [38]. Thus, setting $\mathbf{g} = \mathbf{a}^*$ and $\mathbf{f} = \mathbf{b}^*$, we have

$$(\mathbf{b}^* \times \mathbf{a}^*)^* = \mathbf{a} \times' \mathbf{b}.$$

Q.E.D.

Let $V = \{1,2,\ldots,n\}$, and let $W: V \rightarrow V$ be any function with the property $i \in W(i)$ for all i. Under the induced action of the isomorphism $\Psi^{-1}$ as discussed in Chapter 2, W represents the neighbors of processor $x_i$ with whom processor $x_i$ can communicate. That is,

$\Psi^{-1}(W(i))$ represents those processors in a network whose memories processor $x_i$ can directly access, which includes itself (as $i \in W(i)$). We shall call W a *configuration function* on V.

An example of a configuration function is the von Neumann configuration function. See Figure 1(a). As before, we assume our coordinate set $\mathbf{X}$ is an r x s array, rs = n. Fix $i \in V$. Then $i = k*s + p$, for some non-negative integers k,p where $0 \leq p < s$ (by the Euclidean algorithm for the integers). Let $N(i) = \{ i - 1, i, i + 1, i - s, i + s \}$, for i=1,...,n. Then the von Neumann configuration function W is defined by $W(i) = \{ j \in N(i) : j \in V$, and j satisfies one of the 4 conditions: (1) $j = i$; (2) $j = h*s + p$, where $h = k-1$ or $h = k+1$; (3) if $p \neq 0$, then $j = k*s + 2$; or (4) if $p \neq 1$, then $j = k*s - 1 \}$. This formulation takes care of the truncated von Neumann neighborhood on the boundary pixels of $\mathbf{X}$, as well as for pixels on the interior of $\mathbf{X}$.

We now state some preliminary definitions and concepts, making use of the graph theory developed in section 3.3.3.

Define $\mathcal{S}_{-\infty}(\mathbf{t}_i) = \{ j \in V : t_{ij} \neq -\infty \}$. The set $\mathcal{S}_{-\infty}(\mathbf{t}_i)$ is called the *support* of $\mathbf{t}_i$, in accordance with the image algebra definition of the support of a template at location $\mathbf{x}_i$. In fact, $\Psi^{-1}(\mathcal{S}_{-\infty}(\mathbf{t}_i)) = \mathcal{S}_{-\infty}(\mathbf{t}'_{\mathbf{x}_i})\ \forall\, i$. Let W be a configuration function, and let $\mathbf{t} \in \mathcal{M}_{nn}$. We say $\mathbf{t}$ is *local with respect to* W if $\mathcal{S}_{-\infty}(\mathbf{t}_i) \subset W(i)$ for all $i \in V$. If W is understood we say that $\mathbf{t}$ is local. A *decomposition* of $\mathbf{t}$ is a set of matrices $\{\mathbf{t}(i)\}_{i=1}^{j}$ such that $\mathbf{t} = \bigtimes_{i=1}^{j} \mathbf{t}(i)$. The set $\{\mathbf{t}(i)\}_{i=1}^{j}$ is a *local decomposition of* $\mathbf{t} \in \mathcal{M}_{nn}$ *with respect to* W if $\mathbf{t}(i)$ is local with respect to W for all i =1,...,j. As before, if W is understood, we say simply that $\{\mathbf{t}(i)\}_{i=1}^{j}$ a local decomposition of $\mathbf{t}$. A *weak* decomposition is one in which $\vee$ occurs as a template operation. A *weak local decomposition* is a local decomposition that is weak.

Let $D = \{V,E\}$ be a digraph with $u, v \in V$. We say that v *is reachable from* u if there exists a path from u to v in D. If u is reachable from v and v is reachable from u, then we say that the pair (u,v) or (v,u) is *mutually reachable* in D. Note that in a graph G, reachable and mutually reachable are equivalent. A digraph or graph is *strongly connected* if for all pairs $(u,v) \in V \times V$, u is reachable from v.

We now present the correspondence between template configurations and digraphs. For every configuration function W, we can associate a graph and digraph in the following way. For $i \in V$, let $E_i = \{ (j,i) : j \in W(i) \}$, and let $F_i = \{ (i,j) : (j,i) \in E_i \}$. The *digraph of* W, denoted by D(W), is the digraph $\{ V,E \}$ where $E = \bigcup_{i=1}^{n} E_i$. The *graph* of W is denoted by G(W), and is defined to be $G(W) = \{ V,E \}$, where $E = \bigcup_{i=1}^{n} E_i \cup \bigcup_{i=1}^{n} F_i$.

We remark that if W is the von Neumann or the Moore configuration function, then G(W) is strongly connected. These are common types of neighborhood connection schemes used on parallel architectures.

Now, we establish the correspondence between our weighted graph G(W) and a matrix. Recall in Chapter 3, we described a one-one correspondence between a graph and a template $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$. We use the isomorphism $\Psi^{-1}$ to map a template to a matrix, and define the weighted graph in terms of the function $\alpha$ and $\Psi^{-1}$. Specifically, for a matrix $\mathbf{t} = (t_{ij})$, we define the corresponding graph $\Delta(\mathbf{t})$ as

$$\Delta(\mathbf{t}) \equiv \alpha(\Psi^{-1}(\mathbf{t})),$$

and hence $\Delta(\mathbf{t})$ has edge weight $t_{ji}$ for the edge (i,j). Let W be a configuration function, and G(W) its graph. Then any matrix $\mathbf{t}$ associated with G(W) must satisfy $t_{ji} = -\infty$ if (i,j) is not an edge in G(W). We will use this fact in determining local decompositions for an arbitrary matrix $\mathbf{t}$. The reason for the exchange of indices in this correspondence now becomes

clear. If (j,i) $\in$ E then j $\in$ W(i), and local matrices must satisfy the condition that $\mathcal{S}_{-\infty}(t_i) \subset W(i)$. Thus, for a given i $\in$ V, $\{ j \in V : j \in W(i) \}$ represents the possible indices j where $t_{ij}$ need not have value $-\infty$, or, equivalently, the possible processors $x_j$ who can communicate with processor $x_i$.

### 5.1.2. Local Decompositions

A *matrix decomposition* of **t** is a set of matrices **t**(1),..., **t**(j) such that **t** = **t**(1) $\times$ **t**(2) $\times \cdots \times$ **t**(j). The **t**(i) are called the *factors* of the decomposition. We write $\mathbf{t} = \mathop{\times}\limits_{i=1}^{j} \mathbf{t}(i)$ is a decomposition of **t**.

**Lemma 5.3.** *Suppose that* $\mathbf{s} \times \mathbf{t} = \mathbf{r}$ *is a decomposition of* **r**. *Then this decomposition is not unique, and we have* $\hat{\mathbf{s}} \times \hat{\mathbf{t}}$ *is also a decomposition of* **r**, *with*

$$\hat{\mathbf{s}} = \lambda \times \mathbf{s}, \quad \hat{\mathbf{t}} = \lambda^{-1} \times \mathbf{t}$$

*and* $\lambda \in \mathbf{F}$ *is arbitrary.*

**Proof:** Suppose that $\mathbf{s} \times \mathbf{t}$ is a decomposition of **r**, and let $\lambda \in \mathbf{F}$ be arbitrary. Then $\lambda \times \mathbf{s} = \mathbf{s} \times \lambda$, as (in our case) **F** is commutative. This implies that

$$(\lambda \times \mathbf{s}) \times (\lambda^{-1} \times \mathbf{t}) = (\mathbf{s} \times \lambda) \times (\lambda^{-1} \times \mathbf{t})$$

$$= \mathbf{s} \times (\lambda \times \lambda^{-1}) \times \mathbf{t} = \mathbf{s} \times \mathbf{e} \times \mathbf{t} = \mathbf{s} \times \mathbf{t} = \mathbf{r}$$

Thus, setting $\hat{\mathbf{s}} = \lambda \times \mathbf{s}$, and $\hat{\mathbf{t}} = \lambda^{-1} \times \mathbf{t}$, we see that $\hat{\mathbf{s}} \times \hat{\mathbf{t}} = \mathbf{r}$ also.

Q.E.D.

**Lemma 5.4.** *Let* $\mathbf{t} \in \mathcal{M}_{nn}$ *be such that* $t_{ii} \in \mathbf{F}\ \forall\ i = 1,...,n$. *Then* **t** *is equivalent to a matrix* **s** *which has the property that* $\mathbf{s}_{ii} = \phi\ \forall\ i = 1,...,n$.

**Proof:** Let $\mathbf{d} = \mathrm{diag}(t_{11}^{-1}, t_{22}^{-1}, ..., t_{nn}^{-1})$. Note that **d** is invertible and $\mathbf{d}^{-1} = \mathrm{diag}(t_{11}, t_{22}, .., t_{nn})$. Defining $\mathbf{s} = \mathbf{d} \times \mathbf{t}$, we find that in computing $\mathbf{s}_{ii}$,

$$s_{ii} = \bigvee_{k=1}^{n} d_{ik} \times t_{ki}$$

$$= d_{ii} \times t_{ii} = t_{ii}^{-1} \times t_{ii} = \phi.$$

Hence we have

$$\mathbf{s} = \mathbf{d} \times \mathbf{t} = \mathbf{d} \times \mathbf{t} \times \mathbf{e}$$

which implies, since both **d** and **e** are invertible, that

$$\mathbf{t} = \mathbf{d}^{-1} \times \mathbf{s} = \mathbf{d}^{-1} \times \mathbf{s} \times \mathbf{e}$$

for the n x n identity matrix **e**, and, thus, **t** is similar to **s** which has the required form.

Q.E.D.

A matrix $\mathbf{t} \in \mathcal{M}_{nn}$ satisfying $t_{ii} = \phi \ \forall \ i = 1,\dots,n$ is called a $\phi$-diagonal matrix, or $\phi$-diagonal, for convenience of notation.

Since every matrix **t** such that $t_{ii} \in \mathbf{F}$ is equivalent to one which has $\phi$'s on the diagonal, we may use this to our advantage and prove our theorems for this special type of matrix if it is easier to do, and use the property of equivalence to show that the theorems hold in the more general case.

A square matrix $\mathbf{t} \in \mathcal{M}_{nn}$ is said to be *lower triangular* if it satisfies

$$t_{ij} = -\infty \ \text{ if } i < j$$

and *upper triangular* if it satisfies

$$t_{ij} = -\infty \ \text{ if } i > j\,.$$

**Lemma 5.5.** *Let* $\mathbf{t} \in \mathcal{M}_{nn}$ *be* $\phi$*-diagonal. Then* **t** *has a weak decomposition into lower and upper triangular matrices. In particular,* **t** *can be written as*

$$\mathbf{t} = l \vee \mathbf{u}\,,$$

*where* $l$ ($\mathbf{u}$) *is lower (upper) diagonal, and* $l_{ii} = \mathbf{u}_{ii} = \phi$.

**Proof:** Define $l$, $\mathbf{u} \in \mathcal{M}_{nn}$ by

$$l_{ij} = \begin{cases} t_{ij} & \text{if } i \leq j \\ -\infty & \text{otherwise} \end{cases}$$

$$u_{ij} = \begin{cases} t_{ij} & \text{if } i \geq j \\ -\infty & \text{otherwise} \end{cases}.$$

It is easy to see that $l \vee \mathbf{u} = \mathbf{t}$. This is because

$$[l \vee \mathbf{u}]_{ij} = l_{ij} \vee u_{ij} = \begin{cases} t_{ij} \vee -\infty & \text{if } i < j \\ -\infty \vee t_{ij} & \text{if } i > j \\ t_{ii} \vee t_{ii} = \phi & \text{if } i = j \end{cases}$$

which shows that $l \vee \mathbf{u} = \mathbf{t}$.

Q.E.D.

**Corollary 5.6.** *Let* $\mathbf{t} \in \mathcal{M}_{nn}$ *be lower or upper triangular with the property that* $t_{ii} \in \mathbf{F}\ \forall\, i$. *Then* $\mathbf{t}$ *is equivalent to a matrix which is* $\phi$*-diagonal.*

An *off* matrix $\mathbf{b} \in \mathcal{M}_{nn}$ is a matrix which satisfies:

$b_{ii} \in \mathbf{F}\ \forall\, i = 1,\ldots,n$, and $b_{ij} = -\infty$ if $i \neq j$, except for a unique index pair $(i',j')$ such that $b_{i',j'} \in \mathbf{F}$.

If $\mathbf{t} \in \mathcal{M}_{mn}$, then we use the notation $\mathbf{t}_i$ to denote the i-th row of $\mathbf{t}$, and the notation $\mathbf{t}^j$ to denote the j-th column of $\mathbf{t}$. If $\mathbf{a} \in E^n$, then $a_i$ denotes the i-th entry of the vector $\mathbf{a}$. If $\mathbf{t} \in \mathcal{M}_{mn}$, then $t_{ij}$ denotes the entry at location (i,j). Thus, for $\mathbf{t} \in \mathcal{M}_{mn}$, $(\mathbf{t})_j^i = t_{ij}$, and it is trivial to show

**Proposition 5.7.** $(\mathbf{s} \times \mathbf{t})_j^i = \mathbf{s}_i \times \mathbf{t}^j$, $\dashv$.

Let $l \in \mathcal{M}_{nn}$ be lower triangular with all diagonal entries equal to $\phi$. For k=1,...,n, define

$$^{k}\mathbf{c} = \mathbf{e} \vee [\, l^{k} \times (\mathbf{e}^{k})' \,].$$

Thus, $^{k}\mathbf{c}$ has form

$$^{k}\mathbf{c} = \begin{bmatrix} \phi & & & & & & \\ & \cdot & \phi & & -\infty & & \\ & & & \phi & & & \\ & & & l_{k+1,k} & \phi & & \\ & & & \cdot & & & \\ & -\infty & & \cdot & & \cdot & \\ & & & l_{nk} & & -\infty & \phi \end{bmatrix}.$$

**Lemma 5.8.** *If $l \in \mathcal{M}_{nn}$ is lower triangular and $\phi$-diagonal, then*

$$l = {}^{1}\mathbf{c} \times {}^{2}\mathbf{c} \times \cdots \times {}^{n-1}\mathbf{c}.$$

**Proof:** By induction we show that $\mathbf{s} = {}^{1}\mathbf{c} \times {}^{2}\mathbf{c} \times \cdots \times {}^{k}\mathbf{c}$ has form

$$\mathbf{s} = \begin{bmatrix} \phi & & & & & & & \\ l_{21} & \phi & & & & & & \\ \cdot & l_{32} & \cdot & & & -\infty & & \\ \cdot & \cdot & & \phi & & & & \\ \cdot & \cdot & & l_{k+1,k} & \phi & & & \\ \cdot & \cdot & & \cdot & -\infty & \phi & & \\ \cdot & & & \cdot & \cdot & & \cdot & \\ l_{n1} & l_{n2} & \cdot & l_{nk} & \cdot & & -\infty & \phi \end{bmatrix} \tag{5-1}$$

for $1 \leq k \leq n\text{-}1$. It is easily shown that

$$^{1}\mathbf{c} \times {}^{2}\mathbf{c} = \begin{bmatrix} \phi & & & & & & & \\ l_{21} & \phi & & & & & & \\ \cdot & l_{32} & \cdot & & & -\infty & & \\ \cdot & \cdot & & \phi & & & & \\ \cdot & \cdot & & -\infty & \phi & & & \\ \cdot & \cdot & & \cdot & -\infty & \phi & & \\ \cdot & \cdot & & & \cdot & & \cdot & \\ l_{n1} & l_{n2} & -\infty & -\infty & \cdot & & -\infty & \phi \end{bmatrix}.$$

We assume that $\mathbf{s} = {}^{1}\mathbf{c} \times {}^{2}\mathbf{c} \times \cdots \times {}^{k}\mathbf{c}$ has form as in (5-1) for $1 \leq k \leq n\text{-}2$.

Assume the induction step, and consider $\mathbf{s} \times {}^{k+1}\mathbf{c}$.

**Case 1.** $j \neq k+1$. $(\mathbf{s} \times {}^{k+1}\mathbf{c})^j = \mathbf{s} \times ({}^{k+1}\mathbf{c})^j = \mathbf{s} \times \mathbf{e}^j$, as $({}^{k+1}\mathbf{c})^j = \mathbf{e}^j$ for $j \neq k+1$.

Continuing, $\mathbf{s} \times \mathbf{e}^j = \mathbf{s}^j = \begin{cases} l^j & 1 \leq j \leq k \\ \mathbf{e}_j & k+1 < j \leq n \end{cases}$.

**Case 2.** $j = k+1$. $(\mathbf{s} \times {}^{k+1}\mathbf{c})_{i,k+1} = (\mathbf{s} \times {}^{k+1}\mathbf{c})_i^{k+1} = \mathbf{s}_i \times ({}^{k+1}\mathbf{c})^{k+1} =$

$\bigvee_{h=1}^{n} s_{ih} \times {}^{k+1}c_{h,k+1}$. If $i < k+1$, then

$$\bigvee_{h=1}^{n} s_{ih} \times {}^{k+1}c_{h,k+1} = \left[\bigvee_{h=1}^{i} s_{ih} \times {}^{k+1}c_{h,k+1}\right] \vee \left[\bigvee_{h=i+1}^{k+1} s_{ih} \times {}^{k+1}c_{h,k+1}\right] \vee$$

$$\left[\bigvee_{h=k+2}^{n} s_{ih} \times {}^{k+1}c_{h,k+1}\right]$$

$$= \left[\bigvee_{h=1}^{i} l_{ih} \times -\infty\right] \vee \left[\bigvee_{h=i+1}^{k+1} -\infty \times {}^{k+1}c_{h,k+1}\right] \vee \left[\bigvee_{h=k+2}^{n} -\infty \times {}^{k+1}c_{h,k+1}\right] = -\infty.$$

If $i = k+1$, then $\bigvee_{h=1}^{n} s_{ih} \times {}^{k+1}c_{h,k+1} =$

$$\left[\bigvee_{h=1}^{k} s_{k+1,h} \times {}^{k+1}c_{h,k+1}\right] \vee \left[s_{k+1,k+1} \times {}^{k+1}c_{k+1,k+1}\right] \vee \left[\bigvee_{h=k+2}^{n} s_{k+1,h} \times {}^{k+1}c_{h,k+1}\right]$$

$$= \left[\bigvee_{h=1}^{k} l_{k+1,h} \times -\infty\right] \vee \left[\phi \times \phi\right] \vee \left[\bigvee_{h=k+2}^{n} -\infty \times {}^{k+1}c_{h,k+1}\right] = \phi.$$

If $i > k+1$, then $\bigvee_{h=1}^{n} s_{ih} \times {}^{k+1}c_{h,k+1} = \left[\bigvee_{h=1}^{k} s_{ih} \times {}^{k+1}c_{h,k+1}\right] \vee \left[\bigvee_{h=k+1}^{i-1} s_{ih} \times {}^{k+1}c_{h,k+1}\right]$

$$\vee \left[s_{ii} \times {}^{k+1}c_{i,k+1}\right] \vee \left[\bigvee_{h=i+1}^{n} s_{ih} \times {}^{k+1}c_{h,k+1}\right]$$

$$= \left[\bigvee_{h=1}^{k} l_{ih} \times -\infty\right] \vee \left[\bigvee_{h=k+1}^{i-1} -\infty \times {}^{k+1}c_{h,k+1}\right] \vee \left[\phi \times l_{i,k+1}\right] \vee$$

$$\left[\bigvee_{h=i+1}^{n} -\infty \times {}^{k+1}c_{h,k+1}\right] = l_{i,k+1}.$$

Thus, for $j = k+1$, we have

$$(\mathbf{s} \times {}^{k+1}\mathbf{c})^{k+1} = \begin{bmatrix} -\infty \\ \cdot \\ \cdot \\ \cdot \\ -\infty \\ \phi \\ l_{k+2,k+1} \\ \cdot \\ \cdot \\ l_{n,k+1} \end{bmatrix} \quad \text{and hence,}$$

$$\mathbf{s} \times {}^{k+1}\mathbf{c} = \begin{bmatrix} | & | & . & | & | & . & | \\ l^1 & l^2 & . & l^{k+1} & \mathbf{e}^{k+2} & . & \mathbf{e}^n \\ | & | & . & | & | & . & | \end{bmatrix}.$$

Q.E.D.

**Lemma 5.9.** *Let $l \in \mathcal{M}_{nn}$ be lower triangular and $\phi$-diagonal, and let ${}^k\mathbf{c}$ be as above,* $k=1,\dots,n$. *Then*

$${}^k\mathbf{c} = {}^{n,k}\mathbf{c} \times {}^{n-1,k}\mathbf{c} \times \cdots {}^{k+1,k}\mathbf{c}$$

*where*

$${}^{i,k}\mathbf{c} = \begin{bmatrix} \phi & & & & & & \\ & \cdot & & & & & \\ & & \phi & & -\infty & & \\ & & & \phi & & & \\ & & & -\infty & \phi & & \\ & & & \cdot & -\infty & \cdot & \\ & -\infty & & l_{ik} & \cdot & & \cdot \\ & & & -\infty & & -\infty & \phi \end{bmatrix}, \text{ for } i \geq k+1.$$

**Proof:** We use induction to show that ${}^k\mathbf{c} = {}^{n,k}\mathbf{c} \times {}^{n-1,k}\mathbf{c} \times \cdots {}^{k+1,k}\mathbf{c}$ has the required form. It is easily shown that ${}^{k+2,k}\mathbf{c} \times {}^{k+1,k}\mathbf{c}$ has form

$$\begin{bmatrix}
\phi \\
-\infty & \cdot \\
 & & \phi & & & -\infty \\
 & & l_{k+1,k} & \phi \\
 & & l_{k+2,k} & -\infty & \phi \\
 & & -\infty & & & \phi \\
-\infty & & \cdot & & \cdot & & \cdot \\
 & & -\infty & & -\infty & & -\infty & \phi
\end{bmatrix}.$$

Let $\mathbf{s} = {}^{i,k}\mathbf{c} \times {}^{i-1,k}\mathbf{c} \times \cdots {}^{k+1,k}\mathbf{c}$, for $i \leq n\text{-}1$. Here, we assume

$$\mathbf{s}^j = \begin{cases} \mathbf{e}^j & \text{if } j \neq k \\ \begin{bmatrix} -\infty \\ \cdot \\ \phi \\ l_{k+1,k} \\ l_{k+2,k} \\ \cdot \\ l_{i,k} \\ -\infty \\ \cdot \\ -\infty \end{bmatrix} & \text{if } j = k \end{cases}.$$

Then the j-th column of ${}^{i+1,k}\mathbf{c} \times \mathbf{s}$ is $({}^{i+1,k}\mathbf{c} \times \mathbf{s})^j$.

**Case 1.** $j \neq k$. ${}^{i+1,k}\mathbf{c} \times \mathbf{s}^j = {}^{i+1,k}\mathbf{c} \times \mathbf{e}^j = {}^{i+1,k}\mathbf{c}^j = \mathbf{e}^j$.

**Case 2.** $j = k$.

$$ {}^{i+1,k}\mathbf{c}_m \times \mathbf{s}^k = \bigvee_{h=1}^{n} {}^{i+1,k}c_{mh} \times s_{hk}. \tag{5-2}$$

If $m < k$, then (5-2) is ${}^{i+1,k}\mathbf{c}_m \times \mathbf{s}^k = \mathbf{e}_m \times \mathbf{s}^k = -\infty$.

For $k \leq m \leq i$, equation (5-2) gives us ${}^{i+1,k}\mathbf{c}_m \times \mathbf{s}^k = \begin{cases} \phi & \text{if } m = k \\ l_{mk} & \text{if } m = k+1,\ldots,i \end{cases}$.

If $m = i+1$ then ${}^{i+1,k}\mathbf{c}_m \times \mathbf{s}^k = {}^{i+1,k}\mathbf{c}_{i+1} \times \mathbf{s}^k = l_{i+1,k}$.

If $m > i+1$ then ${}^{i+1,k}\mathbf{c}_m \times \mathbf{s}^k = -\infty$.

Thus, the k-th column of ${}^{i+1,k}\mathbf{c} \times \mathbf{s}$ has form

$$\begin{bmatrix} -\infty \\ \cdot \\ \phi \\ l_{k+1,k} \\ \cdot \\ \cdot \\ l_{i+1,k} \\ -\infty \\ \cdot \\ -\infty \end{bmatrix}.$$

Q.E.D.

We now state the main result of this section.

**Theorem 5.10.** *Let* $\mathbf{t} \in \mathcal{M}_{nn}$ *be a doubly-*$\mathbf{F}$*-astic matrix with* $t_{ii} \in \mathbf{F}$ *for all* $i = 1,\ldots,n$, *and* W *an arbitrary configuration function. Then* $\mathbf{t}$ *has a local weak decomposition if and only if* G(W) *is strongly connected. Furthermore, there is at most one weak operation of* $\bigvee$.

We prove Theorem 5.10 in several steps. First we show that strong connectivity is a necessary condition. Then we derive a general decomposition method for a matrix $\mathbf{t}$, and show how elementary matrices play a crucial role in determining that strong connectivity is sufficient. In the proofs we make no distinction between matrices with values in either belt, $\mathbf{R}_{-\infty}$ or $\mathbf{R}_{-\infty}^{+}$, and hence the operations of $+$ or $*$, denoted by the symbol $\times$. This is because we will use the isomorphisms to show the results hold for templates with values in $\mathbf{R}_{-\infty}$ under the operation $\boxed{\vee}$ as well as for templates with values in $\mathbf{R}_{-\infty}^{+}$ under the operation $\circledvee$.

5.1.3. Necessary and Sufficient Conditions.

Our first theorem shows the sufficiency.

**Theorem 5.11.** *If every* $\mathbf{t} \in \mathcal{M}_{nn}$ *has a local decomposition with respect to* W, *then* G(W) *is strongly connected.*

**Proof**: We assume by contradiction that G(W) is not strongly connected, but every $\mathbf{t} \in \mathcal{M}_{nn}$ has a local decomposition with respect to W. Let i,j $\in$ V such that i is not reachable from j. Let

$$D(1) = \{ k \in V : i \text{ is reachable from } k \}$$

$$D(2) = \{ m \in V : m \text{ is reachable from } j \} .$$

Note that $D(1) \cap D(2) = \emptyset$, for otherwise $z \in D(1) \cap D(2)$ implies that i is reachable from z and z is reachable from j, giving i is reachable from j. Let $\mathbf{t}$ be a matrix which is local with respect to W.

First, we show that $t_{km} = -\infty$ if $k \in D(1)$ and $m \in D(2)$. Suppose by way of contradiction that $k \in D(1)$ and $m \in D(2)$ with $t_{km} \neq -\infty$. Then since $\mathbf{t}$ is local with respect to W, $m \in \mathcal{S}_{-\infty}(\mathbf{t}_k)$ implies that $m \in W(k)$. Thus, (m,k) is an edge and hence k is reachable from m. However, i is reachable from k, so i is reachable from m also. Thus, $m \in D(1)$, which is a contradiction, as m is also in D(2). Thus, $t_{km} = -\infty$ if k $\in D(1)$ and $m \in D(2)$.

We now give a matrix which cannot have a local decomposition with respect to W. Define $\mathbf{t} \in \mathcal{M}_{nn}$ and $\mathbf{v} \in \mathbf{F}^{n}_{-\infty}$ by

$$t_{km} = \begin{cases} \phi & \text{if } (k,m) = (i,j) \\ -\infty & \text{otherwise} \end{cases}, \qquad v_k = \begin{cases} \phi & \text{if } k = j \\ -\infty & \text{otherwise} \end{cases}.$$

By assumption, there exists a decomposition of $\mathbf{t}$, $\mathbf{t} = \underset{h=1}{\overset{r}{\times}} \mathbf{t}^i$. At location k, the vector $\mathbf{b} = \mathbf{t} \times \mathbf{v}$ has value

$$b_k = \bigvee_{h=1}^{n} t_{kh} \times v_h = t_{kj} \times v_j = \begin{cases} \phi & \text{if } k = i \\ -\infty & \text{if } k \neq i \end{cases}$$

Let $p \in \{1,\ldots,r\}$. We show by induction on p that for $\mathbf{c}(p) = (\underset{h=1}{\overset{p}{\times}} \mathbf{t}^h) \times \mathbf{v}$, we have $\mathbf{c}_u(p) = -\infty$ for every index $u \in D(1)$. If this is not true for $p = 1$, then there exists an index $u \in D(1)$ such that $\mathbf{c}_u(1) \in \mathbf{F}$, as $\bigvee_{k=1}^{n} \mathbf{t}^1_{uk} \times v_k = \mathbf{c}_u(1)$. (Note that any $\mathbf{t}^k$ cannot have any $+\infty$ values, as then $\mathbf{t}$ would have $+\infty$ values.) Thus, there must exist a pair (u,m) such that $\mathbf{t}^1_{um} \in \mathbf{F}$ and $\mathbf{v}_m \in \mathbf{F}$, which implies that $m = j$, which implies that $m \in D(2)$. However, $m \in D(2)$ implies that $\mathbf{t}^1_{um} = -\infty$ (as $u \in D(1)$), which is a contradiction. Thus, the claim must be true for $\mathbf{c}(1)$. Now for some $p \in \{2,\ldots,r\}$, by induction assume the claim is true for $\mathbf{c}(1),\ldots,\mathbf{c}(p-1)$, and that it is not true for $\mathbf{c}(p)$. Then there exists a $u \in D(1)$ such that $\mathbf{c}_u(p) \in \mathbf{F}$. Therefore there must exist an index m such that $\mathbf{t}^p_{um} \in \mathbf{F}$ and $\mathbf{c}_m(p-1) \in \mathbf{F}$. By the induction hypothesis, $m \notin D(1)$. Since $\mathbf{t}^p$ is local, we know that $m \in \mathcal{S}_{-\infty}(\mathbf{t}_u) \subset W(u)$ implies that u is reachable from m. Also, since $u \in D(1)$, i is reachable from u so $m \in D(1)$, which is a contradiction to the induction hypothesis. Therefore the claim that for $\mathbf{c}(p) = (\underset{h=1}{\overset{p}{\times}} \mathbf{t}^h) \times \mathbf{v}$, we have $\mathbf{c}_u(p) = -\infty$ for every $u \in D(1)$ is true. In particular, it must be true for $\mathbf{c} = \mathbf{c}(r) = \mathbf{t} \times \mathbf{v}$. Thus, $\mathbf{c}_i(p) = \mathbf{b}_i = -\infty$, which contradicts our construction of $\mathbf{c}$. Hence the proof is complete.

Q.E.D.

It can be easily shown that the set of all strictly doubly $\phi$-astic n x n matrices with entries in $\{-\infty, \phi\}$ is isomorphic to the symmetric group $S_n$ [38]. Since every permutation $\sigma$ can be factored into a product of transpositions, every permutation matrix can be factored into a product of matrices corresponding to transpositions. This leads to our next definition of *exchange matrices.*

The *exchange matrix* $\mathbf{p}^{ij} \in \mathcal{M}_{nn}$ is the matrix defined by

$$\mathbf{p}^{ij}_{km} = \begin{cases} \phi & \text{if } k = m \text{ and } k \neq i,\ k \neq j \\ & \text{or if } (k,m) = (i,j) \text{ or } (k,m) = (j,i) \\ -\infty & \text{otherwise} \end{cases}$$

The matrix $\mathbf{p}^{ij}$ corresponds to the transposition $\sigma = (i,j) \in S_n$. Note that $\mathbf{p}^{ij} \times \mathbf{a} = \mathbf{b}$, where

$$b_k = \begin{cases} a_i & \text{if } k = j \\ a_j & \text{if } k = i \\ a_k & a_k \text{ otherwise} \end{cases} \quad , \text{ for } k = 1,\ldots,n\ .$$

**Lemma 5.12.** *Let* $(i,j) \in V$, $i \neq j$, *let* $i,j$ *denote a transposition in* $S_n$, *and let* $W$ *be a configuration function. Suppose there exists an* $i$ - $j$ *path in* $G(W)$,

$$i = k_0,\ k_1,\ \ldots, k_m = j\ .$$

*Then the exchange matrix* $\mathbf{p}^{ij}$ *can be written as*

$$\mathbf{p}^{ij} = \mathbf{p}^{ik_1} \times \mathbf{p}^{k_1k_2} \times \cdots \times \mathbf{p}^{k_{m-1}j} \times \mathbf{p}^{k_{m-2}k_{m-1}} \times \cdots \times \mathbf{p}^{k_1k_2} \times \mathbf{p}^{ik_1}.$$

**Proof:** Note that $(i,j) \in S_n$ can be written as

$$(i,j) = (i\,k_1)\circ(k_1\,k_2)\ \cdots\ \circ(k_{m-1}\,j)\circ(k_{m-2}\,k_{m-1})\circ\ \cdots\ \circ(k_1\,k_2)\circ(i\,k_1)\,.$$

In the cyclic notation for the permutation $\sigma = (i,j)$, let $i \rightarrow j$ denote i goes to j. Then we have, for the above permutation, $i \rightarrow k_1$, $k_1 \rightarrow k_2$, ..., $k_{m-1} \rightarrow j$, which implies that $i \rightarrow j$. We also have $k_1 \rightarrow i \rightarrow k_1$; $k_2 \rightarrow k_1 \rightarrow k_2$;...; $k_{m-1} \rightarrow k_{m-2} \rightarrow$

$k_{m-1}$; $j \to k_{m-1} \to k_{m-2} \to \ldots \to k_1 \to i$, which implies that $k_i \to k_i$, $i \neq 0,m$, and $j \to i$. We now use the fact that $S_n$ is isomorphic to the set of all strictly doubly $\phi$-astic matrices, where the permutation $\sigma = (i_1, i_2, \ldots, i_m)$ corresponds to the permutation matrix $\mathbf{p}^{i_1, i_2, \ldots, i_m} = \mathbf{p}^\sigma$ defined by

$$\mathbf{p}^\sigma_{hk} = \begin{cases} \phi & \text{if } (h,k) = (i_r, i_{r+1}),\ r = 1,\ldots,m-1 \\ & \quad \text{or if } (h,k) = (i_m, i_1) \\ -\infty & \text{otherwise} \end{cases}$$

and the product of two permutations $\sigma_1$, $\sigma_2$ corresponds to the matrix product $\mathbf{p}^{\sigma_1} \times \mathbf{p}^{\sigma_1}$. Thus, for $\sigma = (i,j) =$

$(i\,k_1) \circ (k_1 k_2) \cdots \circ (k_{m-1}\, j) \circ (k_{m-2} k_{m-1}) \circ \cdots \circ (k_1 k_2) \circ (i\, k_1)$, we have

$$\mathbf{p}^{ij} = \mathbf{p}^{i k_1} \times \mathbf{p}^{k_1 k_2} \times \cdots \times \mathbf{p}^{k_{m-1} j} \times \mathbf{p}^{k_{m-2} k_{m-1}} \times \cdots \times \mathbf{p}^{k_1 k_2} \times \mathbf{p}^{i k_1}.$$

Q.E.D.

**Lemma 5.13.** *Let* $i,j \in V$, $i \neq j$ *and let* $\mathbf{p}^{ij}$ *be the exchange matrix associated with* $i,j$. *Assume there exists an* i-j *path*

$$i = k_0, k_1, \ldots, k_m = j\ .$$

*Then the exchange matrix* $\mathbf{p}^{ij}$ *has a local decomposition with respect to* W.

**Proof:** If $i \in W(j)$ then we are done. Otherwise, use Lemma 5.12 and write

$$\mathbf{p}^{ij} = \mathbf{p}^{i k_1} \times \mathbf{p}^{k_1 k_2} \times \cdots \times \mathbf{p}^{k_{m-1} j} \times \mathbf{p}^{k_{m-2} k_{m-1}} \times \cdots \times \mathbf{p}^{k_1 k_2} \times \mathbf{p}^{i k_1}.$$

Since $(k_h, k_{h+1})$ is an edge of the graph G(W) for all $h = 0,\ldots,m-1$, we know that $k_h \in W(k_{h+1})$ and also that $k_{h+1} \in W(k_h)$ (as G(W) is a graph), for all $h = 0,\ldots,m-1$.

Also,

$$S_{-\infty}(\mathbf{p}_r^{k_h k_{h+1}}) = \begin{cases} \{k_{h+1}\} & \text{if } r = h \\ \{k_h\} & \text{if } r = h+1 \\ \{r\} & \text{otherwise} \end{cases}$$

and hence $S_{-\infty}(\mathbf{p}_r^{k_h,k_{h+1}}) \subset W(r)$ for all r = 1,...,n and for all h = 0,...,m-1. Thus $\mathbf{p}^{ij}$ is local with respect to W for all h = 0,...,m-1 and hence

$$\mathbf{p}^{ij} = \mathbf{p}^{ik_1} \times \mathbf{p}^{k_1k_2} \times \cdots \times \mathbf{p}^{k_{m-1}j} \times \mathbf{p}^{k_{m-2}k_{m-1}} \times \cdots \times \mathbf{p}^{k_1k_2} \times \mathbf{p}^{ik_1}$$

is a local decomposition of $\mathbf{p}^{ij}$ with respect to W.

Q.E.D.

Exchange matrices play an important role in the decomposition method, as we shall see shortly. We now present a decomposition method for lower triangular matrices.

The matrices ${}^k\mathbf{c}$ as described in Lemma 5.8 are fairly sparse, but certainly not local with respect to most architectures. Lemma 5.9 showed that each matrix ${}^k\mathbf{c}$ can be written as a product of even sparser matrices, and in the next theorem we show that these very sparse matrices can be decomposed locally. The matrices ${}^{ik}\mathbf{c}$ (i > k) are off matrices, and each is equivalent to a matrix which is local with respect to W, where G(W) is strongly connected.

**Theorem 5.14.** *Let* $\mathbf{b} \in \mathcal{M}_{nn}$ *be a lower triangular off matrix with off-diagonal entry* $\beta$ *at location* (i,j). *Let* W *be a configuration function such that* G(W) *is strongly connected. Then* $\mathbf{b}$ *is equivalent to a matrix which is local with respect to* W. *Furthermore,* $\mathbf{b}$ *has a local decomposition with respect to* W.

**Proof:** If $j \in W(i)$ then $\mathbf{b}$ is already local with respect to W. Otherwise, $W(i) \supseteq \{ i \}$, and then we can find $k \in V$ such that $k \in W(i)$ ($k \neq i$), assuming without loss of generality that k > i. Let $\mathbf{q}^{kj} = \mathbf{p}^{kj}$. Then the matrix $\mathbf{s}$ defined by

$$\mathbf{s} = \mathbf{p}^{kj} \times \mathbf{b} \times \mathbf{q}^{kj}$$

is a matrix which is local with respect to W. To see this, note that $\mathbf{b} \times \mathbf{q}^{kj}$ exchanges columns j and k of matrix $\mathbf{b}$. Thus,

$$(\mathbf{b} \times \mathbf{q}^{kj})_{hm} = \begin{cases} \phi & \text{if h = m, (h,m)} \neq \text{(j,j) or (h,m)} \neq \text{(k,k)} \\ & \text{or if (h,m) = (j,k) or (h,m) = (k,j)} \\ \beta & \text{if (h,m) = (i,k)} \\ -\infty & \text{otherwise} \end{cases} .$$

The matrix $\mathbf{s} = \mathbf{p}^{kj} \times (\mathbf{b} \times \mathbf{q}^{kj})$ exchanges rows j and k of matrix $\mathbf{b} \times \mathbf{q}^{kj}$, so

$$(\mathbf{p}^{kj} \times (\mathbf{b} \times \mathbf{q}^{kj}))_{hm} = \begin{cases} \phi & \text{if h = m} \\ \beta & \text{if (h,m) = (i,k)} . \\ -\infty & \text{otherwise} \end{cases}$$

Note that $\mathcal{S}_{-\infty}(\mathbf{s}_i) = \{i,k\}$, and $\mathcal{S}_{-\infty}(\mathbf{s}_h) = \{h\}$ for $h \neq i$. Since $\{ i,k \} \subset W(i)$, we have $\mathcal{S}_{-\infty}(\mathbf{s}_i) \subset W(i)$ and hence $\mathbf{s}$ is local with respect to W. Note also that $\mathbf{s}$ (in this case) is no longer lower triangular, as $k > i$.

A similar proof holds for the case $k < i$.

To show that $\mathbf{b}$ has a local decomposition, we write

$$\mathbf{b} = \mathbf{p}^{kj} \times \mathbf{s} \times \mathbf{q}^{kj} ,$$

as $(\mathbf{p}^{h,m})^{-1} = \mathbf{p}^{h,m}$ for any $h,m \in V$. Since G(W) is strongly connected, there exists a j-k path for all pairs $(j,k) \in V$, and hence Lemma 5.13 applies to $\mathbf{p}^{jk}$. Thus $\mathbf{p}^{jk}$ has a local decomposition, and since $\mathbf{s}$ is local and $\mathbf{q}^{jk} = \mathbf{p}^{jk}$, we have a local decomposition for $\mathbf{b}$.

Q.E.D.

We are now in a position to prove the following general theorem.

**Theorem 5.15.** *Let $l \in \mathcal{M}_{nn}$ be a lower triangular matrix with $l_{ii} \neq -\infty$ for all i=1,...,n and W a configuration function such that G(W) is strongly connected. Then $l$ has a local decomposition with respect to W.*

**Proof:** By Corollary 5.6, $l$ is equivalent to a matrix **s** where $c_{ii} = \phi$ for all i:

$$l = \mathbf{d} \times \mathbf{c} \times \mathbf{e} = \mathbf{d} \times \mathbf{c}$$

where **d** is a diagonal matrix. By Lemma 5.8, **c** can be written as

$$\mathbf{c} = {}^{1}\mathbf{c} \times {}^{2}\mathbf{c} \times \cdots \times {}^{n-1}\mathbf{c}.$$

By Lemma 5.9, each ${}^{k}\mathbf{c}$ can be written as

$${}^{k}\mathbf{c} = {}^{n,k}\mathbf{c} \times {}^{n-1,k}\mathbf{c} \times \cdots {}^{k+1,k}\mathbf{c}$$

where the ${}^{i,k}\mathbf{c}$ are as in the statement of Lemma 5.9. Since each ${}^{i,k}\mathbf{c}$ is an off matrix, we have, by Theorem 5.14, a local decomposition with respect to W for each ${}^{i,k}\mathbf{c}$:

$${}^{i,k}\mathbf{c} = \mathop{\times}_{j=1}^{m(i,k)} \mathbf{s}(i,k,j)$$

where the **s**(i,k,j) are the factors of the decomposition, and each **s**(i,k,j) is local with respect to W, and the number of factors m(i,k) is dependent on the values i and k. Thus,

$$l = \mathbf{d} \times \mathbf{c} = \mathbf{d} \times [\, {}^{1}\mathbf{c} \times {}^{2}\mathbf{c} \times \cdots \times {}^{n-1}\mathbf{c} \,]$$

$$= \mathbf{d} \times [\, {}^{n1}\mathbf{c} \times \cdots {}^{21}\mathbf{c} \,] \times$$

$$[\, {}^{n2}\mathbf{c} \times \cdots {}^{32}\mathbf{c} \,] \times \quad \ldots \times {}^{n,n-1}\mathbf{c}$$

$$= \mathbf{d} \times [\, \mathop{\times}_{j=1}^{m(n,1)} \mathbf{s}(n,1,j) \times \cdots \times \mathop{\times}_{j=1}^{m(2,1)} \mathbf{s}(2,1,j) \,] \times \quad \ldots \times \mathop{\times}_{j=1}^{m(n,n-1)} \mathbf{s}(n,n-1,j)$$

where **s**(i,k,j) is local with respect to W for all i,k,j.

Q.E.D.

Using the property that $(\mathbf{s} \times \mathbf{t})' = \mathbf{t}' \times \mathbf{s}'$ as stated in Theorem 5.1, we can prove the same sequence of theorems for an upper triangular matrix **u** satisfying $u_{ii} \neq -\infty$ for all i = 1,...,n. Thus we have

**Theorem 5.16.** *Let* $\mathbf{u} \in \mathcal{M}_{nn}$ *be an upper triangular matrix with* $u_{ii} \in \mathbf{F}$ *for all i, and* W *a configuration function such that* G(W) *is strongly connected. Then* **u** *has a local decomposition with respect to* W.

This leads immediately to the main theorem of this chapter.

**Theorem 5.10.** *Let* $\mathbf{t} \in \mathcal{M}_{nn}$ *be a doubly-*$\mathbf{F}$*-astic matrix with* $t_{ii} \in \mathbf{F}$ *for all* $i = 1,\dots,n$*, and let* W *be an arbitrary configuration function. Then* **t** *has a weak local decomposition with respect to* W *if and only if* G(W) *is strongly connected. Furthermore, there is at most one weak operation of* $\bigvee$.

**Proof:** By Lemma 5.4, we know an arbitrary matrix **t** can be written in form

$$\mathbf{t} = \mathbf{d} \times \mathbf{s},$$

where $s_{ii} = \phi$ for all i and **d** is a diagonal matrix. By Lemma 5.5, $\mathbf{s} = l \vee \mathbf{u}$ where $l$ and **u** are lower and upper triangular matrices, respectively, with $l_{ii} = u_{ii} = \phi$ for all i. Thus,

$$\mathbf{t} = \mathbf{d} \times (l \vee \mathbf{u}).$$

Suppose that G(W) is strongly connected. By Theorem 5.15, $l$ has a local decomposition $l = \mathop{\times}\limits_{j=1}^{k} \mathbf{c}(j)$, and by Theorem 5.16, **u** has a local decomposition $\mathbf{u} = \mathop{\times}\limits_{i=1}^{m} \mathbf{r}(i)$.

Hence, the expression

$$\mathbf{d} \times \left\{ \left[ \mathop{\times}\limits_{j=1}^{k} \mathbf{c}(j) \right] \vee \left[ \mathop{\times}\limits_{i=1}^{m} \mathbf{r}(i) \right] \right\}$$

is a weak decomposition of $\mathbf{d} \times [\, l \vee \mathbf{u} \,]$, and, since each $\mathbf{c}(j)$ and $\mathbf{r}(i)$ is local with respect to W, $\mathbf{t} = \mathbf{d} \times [\, l \vee \mathbf{u} \,] =$

$$\mathbf{d} \times \left\{ \left[ \mathop{\times}\limits_{j=1}^{k} \mathbf{c}(j) \right] \vee \left[ \mathop{\times}\limits_{i=1}^{m} \mathbf{r}(i) \right] \right\}$$

is a weak local decomposition of $\mathbf{t}$.

Theorem 5.11 shows the sufficiency of the statement.

Q.E.D.

As mentioned earlier, this is by no means the most efficient method of decomposition, and that one would not implement the decomposition by following the constructive steps used to prove Theorem 5.10. The results and hypothesis of the theorem are analogous to conditions for existence of solutions of differential equations. Thus, although the results guarantee the existence of local decompositions, efficient methods for computing them must still be developed.

## 5.2. Decomposition of Templates

We now present the results without proof of the previous section in context of the image algebra, using the isomorphism $\Psi$ as defined in Chapter 2. As before, we let V = { 1,...,n } and W be a configuration function on V. We assume that $\mathbf{X} \subset \mathbf{Z}^2$ is finite and rectangular, of size h x k, hk = n, with the lexicographical ordering as set out in Chapter 2, and that $\mathbf{F}_{-\infty}$ is a sub-bounded l-group of $\mathbf{R}_{-\infty}$ or $\mathbf{R}_{-\infty}^{+}$. Everything in section 5.1 applies to templates under either $\boxed{\vee}$ or $\textcircled{\vee}$. We will state the results using the symbol $\boxed{\vee}$ with the understanding that in this section, $\boxed{\vee}$ may be replaced everywhere by $\textcircled{\vee}$ and $+$ replaced everywhere by $*$, and the results will still be valid.

Let $\mathbf{a} \in \mathbf{F}_{\pm\infty}^{\mathbf{Y}}$ and $\mathbf{b} \in \mathbf{F}_{\pm\infty}^{\mathbf{X}}$ be arbitrary. We define the *outer product* of $\mathbf{a}$ and $\mathbf{b}$ to be an element $\mathbf{t} \in (\mathbf{F}_{\pm\infty}^{\mathbf{X}})^{\mathbf{Y}}$, with gray values of

$$\mathbf{t}_{\mathbf{y}}(\mathbf{x}) = \mathbf{a}(\mathbf{y}) + \mathbf{b}(\mathbf{x}).$$

We denote the outer product of $\mathbf{a}$ and $\mathbf{b}$ by $> \mathbf{a}, \mathbf{b} <$.

Let $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$. A *template decomposition* of $\mathbf{t}$ is a set of templates $\mathbf{t}(1), \ldots, \mathbf{t}(j)$ such that $\mathbf{t} = \mathbf{t}(1) \boxed{\vee} \mathbf{t}(2) \boxed{\vee} \cdots \boxed{\vee} \mathbf{t}(j)$. The $\mathbf{t}(i)$ are called the *factors* of the decomposition. We write $\mathbf{t} = \underset{i=1}{\overset{j}{\boxed{\vee}}} \mathbf{t}(i)$ is a decomposition of $\mathbf{t}$. The decomposition is called *weak* if the operation $\bigvee$ replaces any operation $\boxed{\vee}$ in the decomposition. We say $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ is *local with respect to* W if $\mathcal{S}_{-\infty}(\mathbf{t}_{\mathbf{x}_l}) \subset \{\mathbf{y}_j : j \in W(i)\}$ for all $\mathbf{x}_i \in \mathbf{X}$. A decomposition $\{\mathbf{t}(i)\}_{i=1}^{j}$ of $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ is called a *local decomposition of* $\mathbf{t}$ *with respect to* W if $\mathbf{t}(i)$ is local with respect to W for all $i = 1, \ldots, j$.

**Lemma 5.17.** *Let* $\mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{W}}$, $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{W}})^{\mathbf{Y}}$ *be given. Then* $(\mathbf{s} \boxed{\vee} \mathbf{t})' = \mathbf{t}' \boxed{\vee} \mathbf{s}'$*; the dual operations also satisfy this property.*

**Lemma 5.18.** *Suppose that* $\mathbf{s} \boxed{\vee} \mathbf{t} = \mathbf{r}$ *is a decomposition of* $\mathbf{r}$. *Then this decomposition is not unique, and we have* $\hat{\mathbf{s}} \boxed{\vee} \hat{\mathbf{t}}$ *is also a decomposition of* $\mathbf{r}$ *where*

$$\hat{\mathbf{s}} = \mathbf{s} \boxed{\vee} \lambda, \quad \hat{\mathbf{t}} = -\lambda \boxed{\vee} \mathbf{t}$$

*and* $\lambda \in \mathbf{F}$ *is arbitrary,* $\lambda \in (\mathbf{F}_{\pm\infty}^{\mathbf{W}})^{\mathbf{W}}$.

**Lemma 5.19.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be such that* $\mathbf{t}_{\mathbf{x}}(\mathbf{x}) \in \mathbf{F}\ \forall\, \mathbf{x} \in \mathbf{X}$. *Then* $\mathbf{t}$ *is equivalent to a template* $\mathbf{s}$ *which has the property that* $\mathbf{s}_{\mathbf{x}}(\mathbf{x}) = \phi\ \forall\, \mathbf{x} \in \mathbf{X}$. *In this case, we have*

$$\mathbf{s} = \mathbf{d} \boxed{\vee} \mathbf{t}$$

*where*

$$\mathbf{d} = \operatorname{diag}(-\mathbf{t}_{\mathbf{x}_1}(\mathbf{x}_1), -\mathbf{t}_{\mathbf{x}_2}(\mathbf{x}_2), \ldots, -\mathbf{t}_{\mathbf{x}_n}(\mathbf{x}_n)).$$

If $\mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ satisfies $\mathbf{s}_{\mathbf{x}}(\mathbf{x}) = \phi$ for all $\mathbf{x} \in \mathbf{X}$, then we say that $\mathbf{s}$ is *$\phi$-diagonal*. A template $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ is said to be *lower diagonal* if $\Psi(\mathbf{t})$ is a lower diagonal matrix, and *upper diagonal* if $\Psi(\mathbf{t})$ is an upper diagonal matrix. If $\mathbf{t}$ is lower diagonal, then $\mathbf{t}$ satisfies

$$\mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) = -\infty \text{ if } i < j$$

and if $\mathbf{t}$ is upper diagonal then

$$\mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) = -\infty \text{ if } j < i .$$

**Lemma 5.20.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be* $\phi$*-diagonal. Then* $\mathbf{t}$ *has a weak decomposition into lower and upper triangular templates. In particular,* $\mathbf{t}$ *can be written as*

$$\mathbf{t} = l \vee \mathbf{u} ,$$

*where* $l$ ($\mathbf{u}$) *is lower (upper) diagonal, and* $l_{\mathbf{x}_i}(\mathbf{x}_i) = \mathbf{u}_{\mathbf{x}_i}(\mathbf{x}_i) = \phi$. *Here,*

$$l_{\mathbf{x}_j}(\mathbf{x}_i) = \begin{cases} \mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) & i \leq j \\ -\infty & \text{otherwise} \end{cases} ,$$

$$\mathbf{u}_{\mathbf{x}_j}(\mathbf{x}_i) = \begin{cases} \mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i) & j \leq i \\ -\infty & \text{otherwise} \end{cases} .$$

**Corollary 5.21.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be lower or upper triangular with the property that* $\mathbf{t}_{\mathbf{x}}(\mathbf{x}) \in \mathbf{F}\ \forall\ \mathbf{x} \in \mathbf{X}$. *Then* $\mathbf{t}$ *is equivalent to a template which is* $\phi$*-diagonal.*

A template $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ is called an *off* template if $\Psi(\mathbf{t})$ is an off matrix. The *off-entry value occurring at location* (i,j) is the value $\mathbf{t}_{\mathbf{x}_j}(\mathbf{x}_i)$.

**Lemma 5.22.** *If* $l \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *is lower triangular and* $\phi$*-diagonal. Then*

$$l = {}^{1}\mathbf{r} \boxtimes {}^{2}\mathbf{r} \boxtimes \cdots {}^{n-1}\mathbf{r} ,$$

*where*

$${}^{k}\mathbf{r} = \mathbf{1} \vee [ > \mathbf{l}_{\mathbf{x}_k}, l_{\mathbf{x}_k} < ] ,$$

*and* $>$ , $<$ *denotes the outer product.*

Note that $\mathbf{s} = > \mathbf{l}_{\mathbf{x}_k}, l_{\mathbf{x}_k} < \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ and

$$\mathbf{s}_{\mathbf{x}_h}(\mathbf{x}_m) \;=\; \mathbf{l}_{\mathbf{x}_k}(\mathbf{x}_h) + l_{\mathbf{x}_k}(\mathbf{x}_m) \;=\; \begin{cases} l_{\mathbf{x}_k}(\mathbf{x}_m) & h = k \\ -\infty & h \neq k \end{cases}.$$

Thus, $\mathbf{s}_{\mathbf{x}_h} = -\infty$ (the null image on $\mathbf{X}$), if $h \neq k$, and $\mathbf{s}_{\mathbf{x}_h} = l_{\mathbf{x}_k}$ otherwise. Of course, we have $\Psi(\mathbf{s}) = {}^{k}\mathbf{c}$ in this case.

**Lemma 5.23.** *Let* $l \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be lower triangular and* $\phi$*-diagonal, and let* ${}^{k}\mathbf{r}$ *be as in Lemma* 5.22, k=1,...,n. *Then*

$${}^{k}\mathbf{r} \;=\; {}^{n,k}\mathbf{r} \boxtimes {}^{n-1,k}\mathbf{r} \boxtimes \cdots {}^{k+1,k}\mathbf{r},$$

*where* ${}^{i,k}\mathbf{r} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *is defined to be*

$${}^{i,k}\mathbf{r}_{\mathbf{y}}(\mathbf{x}) \;=\; \begin{cases} \phi & \mathbf{y} = \mathbf{x} \\ l_{\mathbf{x}_k}(\mathbf{x}_i) & \mathbf{y} = \mathbf{x}_k,\ \mathbf{x} = \mathbf{x}_i \\ -\infty & \text{otherwise} \end{cases}$$

*for* $i = k+1,\ldots,n$.

Note that each template ${}^{i,k}\mathbf{r}$ is an off template, with off-entry value occurring at location (i,k).

The main result of this chapter is

**Theorem 5.24.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be a doubly-*$\mathbf{F}$*-astic template with* $\mathbf{t}_{\mathbf{x}}(\mathbf{x}) \in \mathbf{F}$ *for all* $\mathbf{x} \in \mathbf{X}$, *and* W *a configuration function. Then* $\mathbf{t}$ *has a weak local decomposition if and only if* G(W) *is strongly connected. Furthermore, there is at most one weak operation of* $\bigvee$.

The sufficiency of Theorem 5.24 is shown next in Theorem 5.25.

**Theorem 5.25.** *If every* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *has a local decomposition with respect to* W, *then* G(W) *is strongly connected.*

The *exchange template* $\mathbf{p}^{\mathbf{x}_i\mathbf{x}_j} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *associated with* (i,j) is the template defined by

$$\mathbf{p}_{\mathbf{z}}^{\mathbf{x}_i\mathbf{x}_j}(\mathbf{u}) = \begin{cases} \phi & \text{if } \mathbf{z} = \mathbf{u} \text{ and } \mathbf{z} \neq \mathbf{x}_i,\ \mathbf{z} \neq \mathbf{x}_j \\ & \text{or if } (\mathbf{u},\mathbf{z}) = (\mathbf{x}_i,\mathbf{x}_j) \text{ or } (\mathbf{u},\mathbf{z}) = (\mathbf{x}_j,\mathbf{x}_i)\ . \\ -\infty & \text{otherwise} \end{cases}$$

The template $\mathbf{p}^{\mathbf{x}_i\mathbf{x}_j}$ corresponds to the transposition $\sigma = (i,j) \in S_n$ and to the exchange matrix $p^{ij}$. Note that $\mathbf{a} \boxtimes \mathbf{p}^{\mathbf{x}_i\mathbf{x}_j} = \mathbf{b}$, where

$$\mathbf{b}(\mathbf{z}) = \begin{cases} \mathbf{a}(\mathbf{x}_i) & \text{if } \mathbf{z} = \mathbf{x}_j \\ \mathbf{a}(\mathbf{x}_j) & \text{if } \mathbf{z} = \mathbf{x}_i \\ \mathbf{a}(\mathbf{x}_k) & \text{otherwise} \end{cases} .$$

**Lemma 5.26.** *Let* $\mathbf{x}_i, \mathbf{x}_j \in \mathbf{X}$, $i \neq j$, *and let* W *be a configuration function. Suppose there exists an* i - j *path in* G(W),

$$i = k_0,\ k_1,\ \ldots, k_m = j\ .$$

*Then the exchange template* $\mathbf{p}^{\mathbf{x}_i\mathbf{x}_j}$ *can be written as*

$$\mathbf{p}^{\mathbf{x}_i\mathbf{x}_j} = \mathbf{p}^{\mathbf{x}_i\mathbf{x}_{k_1}} \boxtimes \mathbf{p}^{\mathbf{x}_{k_1}\mathbf{x}_{k_2}} \boxtimes \cdots \boxtimes \mathbf{p}^{\mathbf{x}_{k_{m-1}}\mathbf{x}_j} \boxtimes \mathbf{p}^{\mathbf{x}_{k_{m-2}}\mathbf{x}_{k_{m-1}}} \boxtimes \cdots \boxtimes \mathbf{p}^{\mathbf{x}_{k_1}\mathbf{x}_{k_2}} \boxtimes \mathbf{p}^{\mathbf{x}_i\mathbf{x}_{k_1}}.$$

**Lemma 5.27.** *Let* $\mathbf{x}_i, \mathbf{x}_j \in \mathbf{X}$, $i \neq j$, *and let* W *be a configuration function. Let* $\mathbf{p}^{\mathbf{x}_i\mathbf{x}_j}$ *be the exchange template associated with* i,j. *Assume there exists an* i-j *path*

$$i = k_0,\ k_1,\ \ldots, k_m = j\ .$$

*Then the exchange template* $\mathbf{p}^{\mathbf{x}_i\mathbf{x}_j}$ *has a local decomposition with respect to* W.

We now present a local decomposition method for an off template.

**Theorem 5.28.** *Let* $\mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be a lower triangular,* $\phi$*-diagonal template with off-entry value of* $\beta$ *at location* (i,j):

$$\mathbf{s}_{\mathbf{x}_k}(\mathbf{x}_m) = \begin{cases} \phi & k = m \\ \beta & (k,m) = (j,i)\ . \\ -\infty & \text{otherwise} \end{cases}$$

*Let* W *be configuration function such that* G(W) *is strongly connected. Then* **s** *is equivalent to a template which is local with respect to* W. *Furthermore,* **s** *has a local decomposition with respect to* W.

**Theorem 5.29.** *Let* $l \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be a lower triangular template with* $l_{\mathbf{x}}(\mathbf{x}) \in \mathbf{F}$ *for all* $\mathbf{x} \in \mathbf{X}$, *and* W *a configuration function such that* G(W) *is strongly connected. Then* $l$ *has a local decomposition with respect to* W.

Using the property of transpose as stated in Theorem 5.17, we can prove theorems 5.19, 5.21, 5.22, 5.23, 5.28 and 5.29 for upper triangular templates.

**Theorem 5.30.** *Let* $\mathbf{u} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be an upper triangular template with* $\mathbf{u}_{\mathbf{x}_i}(\mathbf{x}_i) \in \mathbf{F}$ *for all* $i$, *and* W *a configuration function such that* G(W) *is strongly connected. Then* **u** *has a local decomposition with respect to* W.

The main theorem follows immediately.

**Theorem 5.24.** *Let* $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be a doubly-*$\mathbf{F}$*-astic template with* $\mathbf{t}_{\mathbf{x}}(\mathbf{x}) \in \mathbf{F}$ *for all* $\mathbf{x} \in \mathbf{X}$, *and* W *a configuration function. Then* **t** *has a weak local decomposition if and only if* G(W) *is strongly connected. Furthermore, there is at most one weak operation of* $\bigvee$.

The use of this theorem is made clear by the following discussion. Suppose a lattice transform **t** is to be mapped to a parallel architecture through a decomposition technique, whereby

$$\mathbf{t} = \mathop{\boxtimes}_{i=1}^{h} \mathbf{r}_i \vee \mathop{\boxtimes}_{j=1}^{k} \mathbf{s}_j .$$

Applying the transform **t** to an image **a**, we have

$$\mathbf{a} \boxtimes \mathbf{t} = \mathbf{a} \boxtimes \left[ \mathop{\boxtimes}_{i=1}^{h} \mathbf{r}_i \vee \mathop{\boxtimes}_{j=1}^{k} \mathbf{s}_j \right] = \left[ \mathbf{a} \boxtimes \left[ \mathop{\boxtimes}_{i=1}^{h} \mathbf{r}_i \right] \right] \vee \left[ \mathbf{a} \boxtimes \left[ \mathop{\boxtimes}_{j=1}^{k} \mathbf{s}_j \right] \right]$$

$$= \Big[ ( \cdots ((\mathbf{a} \boxtimes \mathbf{r}_1) \boxtimes \mathbf{r}_2) \cdots ) \boxtimes \mathbf{r}_h \Big] \vee \Big[ ( \cdots ((\mathbf{a} \boxtimes \mathbf{s}_1) \boxtimes \mathbf{s}_2) \cdots ) \boxtimes \mathbf{s}_k \Big].$$

Every $\mathbf{r}_i$, $\mathbf{s}_j$ is local with respect to the network of processors and hence directly implementable on the parallel architecture. The one operation of maximum is related to storage. When computing the transform $\mathbf{a} \boxtimes \mathbf{t}$, the image $\Big[ ( \cdots ((\mathbf{a} \boxtimes \mathbf{r}_1) \boxtimes \mathbf{r}_2) \cdots ) \boxtimes \mathbf{r}_h \Big]$ must be computed, the result stored, the image $\Big[ ( \cdots ((\mathbf{a} \boxtimes \mathbf{s}_1) \boxtimes \mathbf{s}_2) \cdots ) \boxtimes \mathbf{s}_k \Big]$ computed, and then the maximum between the two results taken. Most processing elements in parallel architectures have a small amount of local memory available. If they do not, then the host machine, or another computer which is able to communicate with the parallel one, must store the results separately then return both to the parallel machine to compute the last pointwise maximum. Thus the one operation of maximum is not an unreasonable stipulation in a decomposition.

## 5.3. Applications to Rectangular Templates

A special class of templates within the invariant ones are *rectangular* templates. Let $\mathbf{X} \subset \mathbf{Z}^2$, $|\mathbf{X}| = m$, $|\mathbf{Y}| = n$. Then a *rectangular template* $\mathbf{t} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}$ has a rectangular array for its support (away from the boundary of $\mathbf{X}$). Specifically, $\mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y})$ is a rectangle, for some $\mathbf{y} \in \mathbf{X}$ such that $\mathcal{S}_{-\infty}(\mathbf{t}_\mathbf{y}) \cap \delta\mathbf{X} = \emptyset$, where $\delta\mathbf{X} = \{(i,j) : i=0$ or $m-1$, or $j=0$ or $n-1\}$. We will use the fact that invariant templates correspond to matrices that are block toeplitz with toeplitz blocks [32]. First, we derive the results in matrix notation, and use the isomorphism to map the theorems to image algebra notation. We give conditions under which rectangular templates can be decomposed into the product of two strip templates, one vertical and one horizontal. Since the proof of this theorem is constructive, a decomposition is given. This theorem was presented in its original form by Li [60].

5.3.1. Decomposition of block toeplitz matrices

This section presents results on a decomposition technique for block toeplitz matrices with toeplitz blocks. Matrices that are block toeplitz with toeplitz blocks correspond to invariant templates. Conditions are given to guarantee a decomposition of such matrices into a product of two matrices, each of which correspond to a *strip* template in the image algebra. Strip templates are templates whose supports are a 1 x k array (horizontal), or a k x 1 array (vertical). Even though these decompositions may not be local ones, a decomposition of this type will reduce the number of computations per pixel necessary to compute the transform, on either a parallel or sequential machine. As an example, suppose $\mathbf{r} \in (\mathbf{F}^{\mathbf{X}})^{\mathbf{X}}$ is a rectangular template and has a h x k support. Suppose there exists two templates $\mathbf{t}$ and $\mathbf{s}$ such that $\mathbf{r} = \mathbf{s} \boxed{\vee} \mathbf{t}$, where $\mathbf{t}$ is a rectangular template with a h x 1 support and $\mathbf{s}$ is a rectangular template with a 1 x k support. Then for $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$ and using the associative property of the $\boxed{\vee}$ operation, we have

$$\mathbf{a} \boxed{\vee} \mathbf{r} = \mathbf{a} \boxed{\vee} (\mathbf{s} \boxed{\vee} \mathbf{t}) = (\mathbf{a} \boxed{\vee} \mathbf{s}) \boxed{\vee} \mathbf{t}.$$

Computing $\mathbf{a} \oplus \mathbf{r}$ directly involves finding the maximum of hk additions, while computing $(\mathbf{a} \boxed{\vee} \mathbf{s}) \boxed{\vee} \mathbf{t}$ involves finding the maximum of h + k additions. Thus savings in computation can be realized on sequential as well as parallel processors.

Minimax matrix results. We shall first prove the following lemma.

**Lemma 5.31.** *Suppose that* $\mathbf{s} \in \mathcal{M}_{nn}$ *is a diagonal matrix of form* $\mathbf{s} = \mathrm{diag}(\alpha, ..., \alpha)$, *where* $\alpha \in \mathbf{F}$. *Then for* $\mathbf{t} \in \mathcal{M}_{nn}$, $\mathbf{s} \times \mathbf{t} = \mathbf{t} \times \mathbf{s}$.

**Proof:** Let $\mathbf{r} = \mathbf{s} \times \mathbf{t}$, and $\mathbf{w} = \mathbf{t} \times \mathbf{s}$. Then

$$r_{ij} = \bigvee_{k=1}^{n} s_{ik} \times t_{kj} = s_{ij} \times t_{jj}$$

and

$$w_{ij} = \bigvee_{k=1}^{n} t_{ik} \times s_{kj} \ = t_{ii} \times s_{ij} .$$

Since $t_{ii} = t_{jj} = \alpha \ \forall$ i,j = 1,...,n, we have

$$r_{ij} = s_{ij} \times t_{jj} = s_{ij} \times t_{ii} = t_{ii} \times s_{ij} = w_{ij} .$$

Q.E.D.

**Lemma 5.32.** *Suppose* $\mathbf{s}, \mathbf{t} \in \mathcal{M}_{nn}$ *are each block toeplitz with toeplitz blocks. Denote* $\mathbf{s}$ *and* $\mathbf{t}$ *by*

$$\mathbf{s} = \begin{bmatrix} \sigma^{11} & . & . & . & \sigma^{1h} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma^{h1} & . & . & . & \sigma^{hh} \end{bmatrix} \qquad \mathbf{t} = \begin{bmatrix} \tau^{11} & . & . & . & \tau^{1h} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \tau^{h1} & . & . & . & \tau^{hh} \end{bmatrix}.$$

*Suppose* $\mathbf{s}$ *and* $\mathbf{t}$ *satisfy the following conditions:*

(1) n = hk, and each block $\sigma^{ij}$, $\tau^{ij}$ is a k x k matrix;

(2) there exist 2h-1 constants $\lambda_{ij}$ such that $\tau^{ij} = \text{diag}(\lambda_{ij}, \ldots, \lambda_{ij})$ and the matrix $\Lambda = (\lambda_{ij}) \in \mathcal{M}_{hh}$ is toeplitz;

(3) $\sigma^{ij} = \begin{cases} \Phi_k & i \neq j \\ \mathbf{a} \in \mathcal{M}_{hh} & i = j \end{cases}$, where $\Phi_k$ denotes the null matrix of size k x k, and $\mathbf{a}$ is a toeplitz matrix where $a_{ij} \in \mathbf{F}$;

*If* (1) - (3) *are satisfied, then* $\mathbf{s} \times \mathbf{t} = \mathbf{t} \times \mathbf{s}$ *and is of form*

$$\begin{bmatrix} \sigma^{11}\times \tau^{11} & \sigma^{11}\times \tau^{12} & . & . & \sigma^{11}\times \tau^{1h} \\ \sigma^{22}\times \tau^{21} & \sigma^{22}\times \tau^{22} & . & . & \sigma^{22}\times \tau^{2h} \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma^{hh}\times \tau^{h1} & . & . & . & \sigma^{hh}\times \tau^{hh} \end{bmatrix}.$$

**Proof:** Let $\mathbf{u} = \mathbf{s} \times \mathbf{t}$, and denote the matrix $\mathbf{u} = (u_{ij})$ in block notation by

$$\mathbf{u} = \begin{bmatrix} \mu^{11} & . & . & . & \mu^{1h} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \mu^{h1} & . & . & . & \mu^{hh} \end{bmatrix}.$$

Then

$$u_{ij} = \bigvee_{m=1}^{n} s_{im} + t_{mk} .$$

For i,k $\in Z^+$ and using the division algorithm for integers, we have

$$i = k \cdot p_i + r_i \quad \text{for some } 0 \leq p_i \leq h \text{ and } 0 \leq r_i \leq k-1 .$$

If $r_i = 0$ then write $i = k \cdot (p_i - 1) + k$, where $0 \leq p_i \leq h-1$. Thus, (abusing notation) in this case, we can always write

$$i = k \cdot p_i + r_i \quad \text{for some } 0 \leq p_i \leq h-1, \text{ and } 0 < r_i \leq k .$$

Similarly, we have

$$j = k \cdot p_j + r_j \quad \text{for some } 0 \leq p_j \leq h-1, \text{ and } 0 < r_j \leq k .$$

The element $u_{ij}$ lies in the $(p_i+1, p_j+1)$-th block of $\mathbf{u}$, and is the $(r_i, r_j)$-th entry in that block. The matrix $\mathbf{s}$ has values of $-\infty$ when m indexes outside the block $\sigma^{p_i+1,p_i+1}$, i.e.,

$$s_{im} = -\infty \text{ for values of m not in } \{w : i-r_i+1 \leq w \leq i-r_i+k\}.$$

Thus, $u_{ij} = \bigvee_{m=i-r_i+1}^{i-r_i+k} s_{im} + t_{mj}$. Since the values $s_{im}$, $i-r_i+1 \leq m \leq i-r_i+k$, lie on the $r_i$-th row of block $\sigma^{p_i+1,p_i+1}$, and the values $t_{mj}$ lie on the $r_j$-th column of block $\tau^{p_i+1,p_j+1}$, the value $u_{ij}$ can be represented by

$$u_{ij} = (\sigma^{p_i+1,p_i+1})_{r_i} \times (\tau^{p_i+1,p_j+1})^{r_j}.$$

This is true for any value of $u_{ij}$ such that $u_{ij}$ lies in block $\mu^{p_i+1,p_j+1}$, i.e.,

$$(\sigma^{p_i+1,p_i+1})_{r_i} \times (\tau^{p_i+1,p_j+1})^{r_j} = (\mu^{p_i+1,p_j+1})_{r_i,r_j}, \quad 1 \leq r_i, r_j \leq k.$$

Thus we have

$$\sigma^{p_i+1,p_i+1} \times \tau^{p_i+1,p_j+1} = \mu^{p_i+1,p_j+1}.$$

Since i,j were arbitrary, this relation holds for each value $1 \leq p_i+1,\ p_j+1 \leq h$.

That is,

$$\sigma^{ii} \times \tau^{ij} = \mu^{ij}, \quad \forall\ i,j = 1,\ldots,n.$$

To show commutativity of the product, we note that for all i,j, $\sigma^{ii}$ is toeplitz with finite elements on the diagonal, and the block $\tau^{ij}$ is a diagonal matrix with a constant value on its diagonal. Thus, we have by Lemma 5.31, for all i,j,

$$\sigma^{ii} \times \tau^{ij} = \tau^{ij} \times \sigma^{ii}.$$

Let $\mathbf{w} = \mathbf{t} \times \mathbf{s}$, $\mathbf{w} = (\omega^{ij})$ in block notation, $i,j = 1,\ldots,h$. We show that $\omega^{ij} = \tau^{ij} \times \sigma^{jj}$. Here, $w_{ij} = \bigvee_{m=1}^{n} t_{im} \times s_{mj}$. As before and using the same method and notation, we have the same restrictions on the indices of $s_{mj}$:

$$s_{mj} = -\infty \text{ for values of m not in } \{w : j-r_j+1 \leq w \leq j-r_j+k\},$$

and the values $s_{mj}$ for $j-r_j+1 \leq m \leq j-r_j+k$ lie on the $r_j$-th column of the block $\sigma^{p_j+1,p_j+1}$. Similarly, the values $t_{im}$ for $j-r_j+1 \leq m \leq j-r_j+k$ lie on the $r_i$-th row of block $\tau^{p_{i+1},p_j+1}$, i.e.,

$$(\tau^{p_{i+1},p_j+1})_{r_i} \times (\sigma^{p_j+1,p_j+1})^{r_j} = (\omega^{p_i+1,p_j+1})_{r_i,r_j},$$

and thus in general we have

$$\tau^{ij} \times \sigma^{jj} = \omega^{ij}.$$

Since $\sigma^{ii} = \sigma^{jj}$ for all i,j, we have

$$\omega^{ij} = \tau^{ij} \times \sigma^{jj} = \tau^{ij} \times \sigma^{ii} = \sigma^{ii} \times \tau^{ij} = \mu^{ij}.$$

Q.E.D.

Let us define a *strip matrix* $\mathbf{s} \in \mathcal{M}_{nn}$ if $\Psi^{-1}(\mathbf{s})$ is a strip template. A *vertical* strip matrix $\mathbf{t} \in \mathcal{M}_{nn}$, $n = hk$, is a matrix which is block toeplitz with toeplitz blocks, $\mathbf{t} = (\tau^{ij})$, each $\tau^{ij}$ is k x k, i,j = 1,...,h, and $\tau^{ij} = \mathrm{diag}(\alpha_{ij}, \ldots, \alpha_{ij})$ for some $\alpha_{ij} \in \mathbf{F}_{-\infty}$. Since $\tau^{ij} = \tau^{1,j-i+1}$ if $j \geq i$ and $\tau^{ij} = \tau^{i-j+1,1}$ if $i \geq j$, there are actually only 2h-1 constants $\alpha_{ij}$ which determine the $\tau^{ij}$. Denote these constants by $\alpha_{11}, \alpha_{i1}, \alpha_{1j}$, i,j = 2,...,h, where

$$\tau^{ij} = \begin{cases} \mathrm{diag}(\alpha_{11}, \ldots, \alpha_{11}) & \text{if } i = j \\ \mathrm{diag}(\alpha_{1j}, \ldots, \alpha_{1j}) & \text{if } j > i \\ \mathrm{diag}(\alpha_{i1}, \ldots, \alpha_{i1}) & \text{if } i > j \end{cases}.$$

Also, we must have $\alpha_{1j} \in \mathbf{F}$ for $j = 1,2,..,m_1$ and $\alpha_{i1} \in \mathbf{F}$ for $j = 1,2,..,m_2$, for some $1 \leq m_1$, $m_2 \leq h$. Vertical strip matrices correspond to vertical strip templates.

A *horizontal* strip matrix $\mathbf{s} \in \mathcal{M}_{nn}$, $n = hk$, is a matrix which is block toeplitz with toeplitz blocks, $\mathbf{s} = (\sigma^{ij})$, each $\sigma^{ij}$ is k x k, i,j = 1,...,h, and $\sigma^{ij} = \begin{cases} \Phi_k & i \neq j \\ \mathbf{a} \in \mathcal{M}_{hh} & i = j \end{cases}$, where $\Phi_k$ denotes the null matrix of size k x k, and $\mathbf{a}$ is a toeplitz matrix where $a_{ii} \in \mathbf{F}$. Horizontal strip matrices correspond to horizontal strip templates. Note that such a matrix is a *diagonal* block toeplitz matrix with toeplitz blocks, that is, the only non-null block is the diagonal block.

We are now in a position to prove the main result of this section.

**Theorem 5.33.** *Assume* $\mathbf{u} \in \mathcal{M}_{nn}$, $n = hk$, *is block toeplitz with toeplitz blocks, and write*

$$\mathbf{u} = \begin{bmatrix} \mu^{11} & . & . & . & \mu^{1h} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \mu^{h1} & . & . & . & \mu^{hh} \end{bmatrix}$$

*where each* $\mu^{ij}$ *is a* k x k *submatrix and* $(\mu^{ii})_{nm} \neq -\infty \ \forall$ i,m. *Then* $\mathbf{u}$ *is decomposable into two strip matrices, one horizontal* $\mathbf{s}$ *and one vertical* $\mathbf{t}$, *if and only if*

(1)$\exists$ 2h-1 constants $\lambda_{1j}$, $\lambda_{i1}$, i,j = 1,...,h with $\lambda_{ii} = 0$, such that $\Lambda = (\lambda_{ij}) \in \mathcal{M}_{hh}$ is toeplitz and of form

$$\Lambda = \begin{bmatrix} 0 & \lambda_{12} & . & . & \lambda_{1k} \\ . & 0 & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \lambda_{k1} & \lambda_{k2} & . & . & 0 \end{bmatrix};$$

(2) the constants $\lambda_{ij}$ satisfy $\lambda_{ij} \times \mu^{ii} = \mu^{ij}$ for all i,j = 1,...,k.

*In this case a decomposition is given by the following matrices which are each block toeplitz with toeplitz blocks:*

$$s = \begin{bmatrix} \sigma^{11} & . & . & . & \sigma^{1h} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma^{h1} & . & . & . & \sigma^{hh} \end{bmatrix} \qquad t = \begin{bmatrix} \tau^{11} & . & . & . & \tau^{1h} \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ \tau^{h1} & . & . & . & \tau^{hh} \end{bmatrix}$$

*where each of* $\sigma^{ij}$, $\tau^{ij}$ *are* k x k *matrices and satisfy:*

$$\tau^{ij} = \mathrm{diag}(\lambda_{ij}, \ldots, \lambda_{ij}); \qquad \sigma^{ij} = \begin{cases} \Phi_k & i \neq j \\ \mu^{ii} \in \mathcal{M}_{hh} & i = j \end{cases}.$$

*Here,* $\lambda_{11} = 0$, *and* $\lambda_{1j} = u_{ij} \times (u_{11})^{-1}$, $\lambda_{i1} = u_{i1} \times (u_{ii})^{-1}$, i,j = 2,...,h.

**Proof:** Suppose (1) and (2) are satisfied. We show that for **s** and **t** as in the statement of the theorem, $\mathbf{s} \times \mathbf{t} = \mathbf{u}$. Let $\mathbf{w} = \mathbf{s} \times \mathbf{t}$. By Lemma 5.32, **w** has form

$$\mathbf{w} = \begin{bmatrix} \sigma^{11} \times \tau^{11} & . & . & . & \sigma^{11} \times \tau^{1h} \\ \sigma^{22} \times \tau^{21} & . & . & . & \sigma^{22} \times \tau^{2h} \\ . & . & . & . & . \\ . & . & . & . & . \\ \sigma^{hh} \times \tau^{h1} & . & . & . & \sigma^{hh} \times \tau^{hh} \end{bmatrix}.$$

Choose an element $w_{ij}$. As in Lemma 5.32,

$$i \;=\; k \cdot p_i + r_i \quad \text{for some } 0 \le p_i \le h-1, \text{ and } 0 < r_i \le k\,.$$

$$j \;=\; k \cdot p_j + r_j \quad \text{for some } 0 \le p_j \le h-1, \text{ and } 0 < r_j \le k\,.$$

and the pair (i,j) lies in the block $(\,p_i + 1,\ p_j + 1\,)$. Thus,

$$w_{ij} \;=\; (s \times t)_{ij} \;=\; (\sigma^{p_i+1,\,p_i+1})_{r_i} \times (\,\tau^{p_i+1,\,p_j+1})^{r_j}$$

$$=\; \bigvee_{m=i-r_i+1}^{i-r_i+k} u_{im} + t_{mj}\,.$$

Since $\tau^{p_i+1,\,p_j+1} = \text{diag}(\,\lambda_{p_i+1,\,p_j+1}, \ldots, \lambda_{p_i+1,p_j+1})$, the only possible non-null element of $\{\,t_{mj} : i-r_i+1 \le i-r_i+k\,\}$ lies on the diagonal of $\tau^{p_i+1,\,p_j+1}$, and, in fact, lies in the $(\,r_j, r_j\,)$-th position in $\tau^{p_i+1,\,p_j+1}$. The $(\,r_j, r_j\,)$-th position in $\tau^{p_i+1,\,p_j+1}$ is in the $(i, i-r_i+r_j)$-th position in $\mathbf{u}$. So

$$w_{ij} \;=\; u_{i,i-r_i+r_j} \times \lambda_{p_i+1,p_j+1}\,.$$

By (1) we know that

$$\lambda_{p_i+1,p_j+1} \times \mu^{p_i+1,\,p_i+1} \;=\; \mu^{p_i+1,\,p_j+1}$$

which implies that

$$(\,\lambda_{p_i+1,p_j+1})_{r_i} \times (\,\mu^{p_i+1,\,p_i+1})^{r_j} \;=\; (\,\mu^{p_i+1,\,p_j+1})_{r_i,r_j} \;=\; u_{ij}$$

which implies

$$\lambda_{p_i+1,p_j+1} \times (\,\mu^{p_i+1,\,p_j\,1+})_{r_i,r_j} \;=\; u_{ij}\,.$$

But

$$(\,\mu^{p_i+1,\,p_j+1})_{r_i,r_j} \;=\; u_{i,i-r_i+r_j}\,.$$

Therefore,

$$u_{ij} \;=\; u_{i,i-r_i+r_j} \times \lambda_{p_i+1,p_j+1} \;=\; w_{ij}\,.$$

Now suppose that $\mathbf{u}$ is decomposable into two strip matrices, a horizontal one $\mathbf{s}$ and a vertical one $\mathbf{t}$. We show that $\mathbf{s}$ and $\mathbf{t}$ satisfy conditions (1) - (3). By the definition of a vertical strip matrix, we see that $\mathbf{t}$ is block toeplitz with toeplitz blocks, and

$\mathbf{t} = \tau^{ij}$ is of form

$$\tau^{ij} \;=\; \begin{cases} \mathrm{diag}(\alpha_{11}, \ldots, \alpha_{11}) & \text{if } i = j \\ \mathrm{diag}(\alpha_{1j}, \ldots, \alpha_{1j}) & \text{if } j > i \\ \mathrm{diag}(\alpha_{i1}, \ldots, \alpha_{i1}) & \text{if } i > j \end{cases}$$

for some 2h-1 constants $\alpha_{11}, \alpha_{i1}, \alpha_{1j}$, i,j = 2,...,h. Set $\alpha_{11} = 0$, $\alpha_{1j} = u_{1j} \times (u_{11})^{-1}$,

$\alpha_{i1} = u_{i1} \times (u_{ii})^{-1}$. Define $\Lambda = (\lambda_{ij}) \in \mathcal{M}_{hh}$ by $\lambda_{ij} = \begin{cases} \alpha_{11} & \text{if } i = j \\ \alpha_{i1} & \text{if } j < i \\ \alpha_{1j} & \text{if } i < j \end{cases}$. Then $\Lambda$ is

toeplitz, with the 2h-1 constants $\lambda_{11}, \lambda_{i1}, \lambda_{1j}$, i,j = 2,...,h, satisfying condition (1).

Also, $\mathbf{s}$ is a horizontal strip matrix, which means that (using block notation)

$\mathbf{s} = (\sigma^{ij})$, where

$$\sigma^{ij} = \begin{cases} \Phi_k & i \neq j \\ \mathbf{a} \in \mathcal{M}_{hh} & i = j \end{cases},$$

and $\mathbf{a}$ is toeplitz with non-null diagonal entries. Set $\mathbf{a} = \mu^{ii}$. The last thing we need to show is that condition (2) is satisfied. By Lemma 5.32, the (i,j)-th block of $\mathbf{s} \times \mathbf{t}$ is of form

$$\sigma^{ii} \times \tau^{ij},$$

and since $\tau^{ij}$ is diagonal, $\sigma^{ii} \times \tau^{ij} = \tau^{ij} \times \sigma^{ii}$. So, $\tau^{ij} \times \sigma^{ii} = \mathrm{diag}(\alpha_{ij} ..., \alpha_{ij}) \times \mu^{ii} = \lambda_{ij} \times \mu^{ii} = \mu^{ij}$. Obviously, $\mathbf{s} \times \mathbf{t} = \mathbf{u}$, and this completes the proof.

Q.E.D.

The statement in the image algebra of Theorem 5.33 is

**Theorem 5.34** [60]. *Let* $\mathbf{r} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be a rectangular template with non-null weights* $\mathbf{r}_{\mathbf{x}_i}(\mathbf{x}_j)$, $i = 1,...,m$, $j = 1,...,k$. *Then* $\mathbf{r}$ *has a decomposition into two strip templates, one horizontal and one vertical, if and only if for all* $1 \leq i,i' \leq m$ *and* $1 \leq j,j' \leq k$,

$$\mathbf{r}_{\mathbf{x}_i}(\mathbf{x}_j) - \mathbf{r}_{\mathbf{x}_{i'}}(\mathbf{x}_j) \;=\; \mathbf{r}_{\mathbf{x}_i}(\mathbf{x}_{j'}) - \mathbf{r}_{\mathbf{x}_{i'}}(\mathbf{x}_{j'}).$$

CHAPTER 6
THE DIVISION ALGORITHM

## 6.1. A Division Algorithm in a Non-Euclidean Domain

The integers have the property that a *division algorithm* can be defined on them. For a,b $\in \mathbf{Z}$, there exist unique integers q,r such that $a = qb + r$ where $|r| < |b|$. This is an example of an integral domain upon which is defined a *Euclidean valuation* [61]. In this section we present a division algorithm for the minimax algebra structure, and give an application of this result to image processing in the image algebra notation.

We remark that the boolean case has already been stated by P. Miller [62], and will be discussed in more detail at the end of this section.

### 6.1.1. A Matrix Division Algorithm

Let $E^1 = \mathbf{F}_{-\infty}$ be a sub-bounded l-group of $\mathbf{R}_{-\infty}$. For notational convenience, we will write $\mathbf{t} \in \mathcal{M}_{nn}(-\infty)$ when we mean that the matrix $\mathbf{t}$ will assume values only on $\mathbf{F} \cup \{-\infty\}$. Similarly, we write $\mathbf{t} \in \mathcal{M}_{nn}(+\infty)$ when the matrix $\mathbf{t}$ assumes values only on $\mathbf{F} \cup \{+\infty\}$. We will show that for a finite vector $\mathbf{a} \in \mathbf{F}^n$ and a subset of matrices of $\mathcal{M}_{nn}(-\infty)$, that there exist vectors $\mathbf{q}$ and $\mathbf{r} \in \mathbf{F}^{n-\infty}$ such that

$$\mathbf{a} = (\mathbf{t}' \times \mathbf{q}) \vee \mathbf{r}$$

**Lemma 6.1.** *Let* $\mathbf{a} \in \mathbf{F}^n$ *be finite, and* $\mathbf{t} \in \mathcal{M}_{nn}(-\infty)$ *satisfy* $\mathcal{S}_{-\infty}(\mathbf{t}_i) \neq \emptyset \; \forall \; i = 1,\ldots,n$. *Define* $\hat{\mathbf{t}}$ *by* $\hat{\mathbf{t}} \equiv (\mathbf{t}^*)'$. *Then both* $\hat{\mathbf{t}} \times' \mathbf{a}$ *and* $\mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a})$ *are finite, and*

$$\mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a}) \leq \mathbf{a}.$$

**Proof:** First we note that

$$(\hat{\mathbf{t}})_{ij} = [(\mathbf{t}^*)']_{ij} = (\mathbf{t}^*)_{ji} = (t_{ij})^* = \begin{cases} -t_{ij} & \text{if } t_{ij} \in \mathbf{F} \\ +\infty & \text{if } t_{ij} = -\infty \end{cases}$$

and that $S_{+\infty}(\hat{\mathbf{t}}_i) = S_{-\infty}(\mathbf{t}_i)$. Let $\mathbf{b} = \hat{\mathbf{t}} \times' \mathbf{a}$ and let $\mathbf{c} = \mathbf{t}' \times \mathbf{b}$. At location i, $b_i = \bigwedge_{j \in S_{+\infty}(\hat{\mathbf{t}}_i)} \hat{t}_{ij} + a_j$. The vector $\mathbf{b}$ is finite, as for $i \in \{1,2,\ldots,n\}$, there exists at least one $j \in S_{+\infty}(\hat{\mathbf{t}}_i)$ such that $\hat{t}_{ij} + a_j$ is finite, since by hypothesis, $S_{-\infty}(\mathbf{t}_i) \neq \emptyset \; \forall$ i $= 1,\ldots,n$, and $a_i \in \mathbf{F} \; \forall \; i = 1,\ldots,n$ also. Thus, $b_i = \left\{ \bigwedge_{j \in S_{+\infty}(\hat{\mathbf{t}}_i)} \hat{t}_{ij} + a_j \right\} \in \mathbf{F} \; \forall$ i. At location i, $c_i = \bigvee_{j \in S_{-\infty}(\mathbf{t}'_i)} t'_{ij} + b_j$. By a similar argument, we see that $c_i \in \mathbf{F} \; \forall$ i.

Suppose that $c_i = t'_{ik} + b_k$ for some $k \in \{1,2,\ldots,n\}$. Then $b_k = \bigwedge_{j \in S_{+\infty}(\hat{\mathbf{t}}_k)} \hat{t}_{kj} + a_j = \hat{t}_{kp} + a_p$ for some p. Since $k \in S_{-\infty}(\mathbf{t}'_i)$, we know that $i \in S_{-\infty}(\mathbf{t}_k)$. Since $S_{-\infty}(\mathbf{t}_k) = S_{+\infty}(\hat{\mathbf{t}}_k)$, we know that $i \in S_{+\infty}(\hat{\mathbf{t}}_k)$, and, hence

$$\hat{t}_{kp} + a_p \leq \hat{t}_{ki} + a_i \in \mathbf{F},$$

by our choice of p and the fact that $\hat{t}_{ki} \in \mathbf{F}$ and $a_i \in \mathbf{F} \; \forall$ i. Thus

$$c_i = t'_{ik} + b_k = t'_{ik} + \hat{t}_{kp} + a_p \leq t'_{ik} + \hat{t}_{ki} + a_i = t_{ki} + (-t_{ki}) + a_i = a_i$$

Thus, $c_i \leq a_i$, and our lemma is proved.

Q.E.D.

We now state the Division Algorithm.

**Theorem 6.2. The Division Algorithm.** *Let* **a**, **t** *satisfy the hypothesis of Lemma* 6.1. *Then for* $\mathbf{q} = \hat{\mathbf{t}} \times' \mathbf{a}$, *and* **r** *defined by*

$$r_i = \begin{cases} a_i & \text{if } a_i > [\mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a})]_i \\ -\infty & \text{if } a_i = [\mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a})]_i \end{cases}$$

*we have*

$$\mathbf{a} = (\ \mathbf{t}' \times\ \mathbf{q}\ ) \vee \mathbf{r}$$

**Proof:** By Lemma 6.1, $\mathbf{a} \geq \mathbf{t}' \times\ \mathbf{q} = \mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})$, and hence,

$$\mathbf{a} \geq \mathbf{r}.$$

Thus, $[\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})] \vee \mathbf{r} \leq \mathbf{a}$. To show that equality holds, that is, that $[\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})] \vee \mathbf{r} = \mathbf{a}$, we examine two cases.

Case 1. $a_i > [\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})]_i$. Here, $[\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})]_i \vee r_i = [\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})]_i \vee a_i = a_i$.

Case 2. $a_i = [\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})]_i$. Here, $[\mathbf{t}' \times\ (\hat{\mathbf{t}} \times' \mathbf{a})]_i \vee r_i = a_i \vee r_i = a_i \vee -\infty = a_i$.

Q.E.D.

Now suppose we have $\mathbf{a} = (\ \mathbf{t}' \times\ \mathbf{q}\ ) \vee \mathbf{r}$ for $\mathbf{a}$ finite, $\mathbf{t} \in \mathcal{M}_{nn}(-\infty)$ and $\mathbf{t}$ satisfying $\mathcal{S}_{-\infty}(\mathbf{t}_i) \neq \emptyset\ \forall\ i = 1,\ldots,n$. Define

$$\mathbf{a}^0 = \mathbf{a}$$

$$\mathbf{r}^0 = \mathbf{r}, \text{ and}$$

$$\mathbf{a}^i = \hat{\mathbf{t}} \times' \mathbf{a}^{i-1}.$$

Then we have

$$\mathbf{a} = \mathbf{a}^0 = (\ \mathbf{t}' \times\ \mathbf{a}^1\ ) \vee \mathbf{r}^0 \tag{6-1}$$

By Lemma 6.1, $\mathbf{a}^1 = \hat{\mathbf{t}} \times' \mathbf{a}^0$ is finite, and, in fact, $\mathbf{a}^i = \hat{\mathbf{t}} \times' \mathbf{a}^{i-1}$ will be finite for each $i = 1,2,\ldots$ Thus, the Division Algorithm applies in particular to $\mathbf{a}^1$:

$$\mathbf{a}^1 = (\ \mathbf{t}' \times\ \mathbf{a}^2\ ) \vee \mathbf{r}^1 \tag{6-2}$$

and substituting (6-2) into (6-1), we get

$$\mathbf{a} = (\ \mathbf{t}' \times\ \mathbf{a}^1\ )\ \vee\ \mathbf{r}^0$$

$$= \{ \mathbf{t}' \times [ ( \mathbf{t}' \times \mathbf{a}^2 ) \vee \mathbf{r}^1 ] \} \vee \mathbf{r}_0$$

$$= ( \mathbf{t}' \times \mathbf{t}' \times \mathbf{a}^2 ) \vee ( \mathbf{t}' \times \mathbf{r}^1 ) \vee \mathbf{r}^0$$

$$= [ (\mathbf{t}')^2 \times \mathbf{a}^2 ] \vee ( \mathbf{t}' \times \mathbf{r}^1 ) \vee \mathbf{r}^0 \tag{6-3}$$

where $(\mathbf{t}')^k$ denotes the k-fold product of $\mathbf{t}$, $\underset{i=1}{\overset{k}{\times}} (\mathbf{t}')$.

Apply the Division Algorithm to $\mathbf{a}^2$, to get

$$\mathbf{a}^2 = ( \mathbf{t}' \times \mathbf{a}^3 ) \vee \mathbf{r}^2$$

and substituting this into (6-3), we get

$$\mathbf{a} = \{ (\mathbf{t}')^2 \times [ ( \mathbf{t}' \times \mathbf{a}^3 ) \vee \mathbf{r}^2 ] \} \vee ( \mathbf{t}' \times \mathbf{r}^1 ) \vee \mathbf{r}^0$$

$$= [ ( \mathbf{t}' )^2 \times \mathbf{t}' \times \mathbf{a}^3 ] \vee [ (\mathbf{t}')^2 \times \mathbf{r}^2 ] \vee [ \mathbf{t}' \times \mathbf{r}^1 ] \vee \mathbf{r}^0$$

$$= [ ( \mathbf{t}' )^3 \times \mathbf{a}^3 ] \vee [ (\mathbf{t}')^2 \times \mathbf{r}^2 ] \vee [ \mathbf{t}' \times \mathbf{r}^1 ] \vee \mathbf{r}^0$$

We can continue like this up to any k-th iteration.

$$\mathbf{a} = \mathbf{r}^0 \vee [ \mathbf{t}' \times \mathbf{r}^1 ] \vee [ (\mathbf{t}')^2 \times \mathbf{r}^2 ] \vee \cdots \vee [ (\mathbf{t}')^k \times \mathbf{r}^k ] \vee [ (\mathbf{t}')^{k+1} \times \mathbf{a}^{k+1} ]$$

or, if we let $(\mathbf{t}')^0$ denote the identity matrix $\mathbf{e}$, we have

$$\mathbf{a} = \bigvee_{i=1}^{k} [ (\mathbf{t}')^i \times \mathbf{r}^i ] \vee [ (\mathbf{t}')^{k+1} \times \mathbf{a}^{k+1} ]$$

We now state a result which will be useful in describing the division algorithm in the image algebra.

**Lemma 6.3.** *Let* $\mathbf{a},\mathbf{b} \in \mathbf{F}^n$ *(be finite vectors). Then we may express the difference of vectors* $\mathbf{a}$ *and* $\mathbf{b}$, a - b, *using the following matrix transform. Define* $\mathbf{s} \in \mathcal{M}_{nn}(-\infty)$ *by* $\mathbf{s} = \mathrm{diag}( (b_1)^*,\ldots,(b_n)^* ) = \mathrm{diag}( -b_1,\ldots,-b_n )$ *with* $-b_i$ *denoting the real arithmetic additive inverse of the real number* $b_i$. *Then*

$$\mathbf{s} \times \mathbf{a} = \mathbf{c} \in \mathbf{F}^n, \quad \text{where}$$

$$c_i = a_i - b_i, \quad i = 1,\ldots,n.$$

**Proof:**

$$(\mathbf{s} \times \mathbf{a})_i = \bigvee_{k=1}^{n} (s_{ik} + a_k) = s_{ii} = a_i = -b_i + a_i$$

for $i = 1,\ldots,n$.

Q.E.D.

We remark that the vector $\mathbf{r}$ as defined in Theorem 6.2 can be obtained in the following way. Fix $\mathbf{a} \in \mathbf{F}^n$, finite. Define $f_{\mathbf{a}} : \mathbf{F}^n \rightarrow \mathbf{F}^n$ by

$$f_{\mathbf{a}}(\mathbf{x}) = \mathbf{y} \text{ where } y_i = \begin{cases} 0 & \text{if } a_i > x_i \\ -\infty & \text{otherwise} \end{cases}.$$

Then for $\mathbf{x} = \mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a})$, $f_{\mathbf{a}}(\mathbf{x}) = \begin{cases} 0 & \text{if } a_i > [\mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a})]_i \\ -\infty & \text{if } a_i \le [\mathbf{t}' \times (\hat{\mathbf{t}} \times' \mathbf{a})]_i \end{cases}$. However, it is easily shown that $f_{\mathbf{a}}$ is not an s-lattice homomorphism. For example, choose $n = 2$, $\mathbf{a}$, $\mathbf{d}$, and $\mathbf{e}$ as below:

$$\mathbf{a} = \begin{bmatrix} 3 \\ -1 \end{bmatrix}, \quad \mathbf{d} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

Then $f_{\mathbf{a}}(\mathbf{d} \vee \mathbf{e}) = f_{\mathbf{a}}\left(\begin{bmatrix} 3 \\ -2 \end{bmatrix}\right) = \begin{bmatrix} -\infty \\ 0 \end{bmatrix}$, but $f_{\mathbf{a}}(\mathbf{d}) \vee f_{\mathbf{a}}(\mathbf{e}) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \vee \begin{bmatrix} -\infty \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \neq f_{\mathbf{a}}(\mathbf{d} \vee \mathbf{e})$.

Thus, according to Theorem 3.1, $f_{\mathbf{a}}$ cannot be represented as a matrix transform. If, however, we go outside of the structure of the minimax algebra, and use image algebra operations in addition to $\vee$ and $\boxtimes$ (or $\circledvee$), we can express this transform in a succinct way, as will be demonstrated in the next section.

<u>A dual division algorithm.</u> The duality of the operations of the matrix algebra enable us to describe a dual division algorithm. We omit the proofs, as they are the dual of the proofs given in the previous section.

**Lemma 6.4.** *Let* $\mathbf{a} \in \mathbf{F}^n$ *be finite, and* $\mathbf{t} \in \mathcal{M}_{nn}(+\infty)$ *satisfy* $\mathcal{S}_{+\infty}(\mathbf{t}_i) \neq \emptyset \; \forall \; i = 1,\ldots,n$. *Define* $\hat{\mathbf{t}}$ by $\hat{\mathbf{t}} \equiv (\mathbf{t}^*)'$. *Then both* $\mathbf{t}' \times \mathbf{a}$ *and* $\hat{\mathbf{t}} \times' (\mathbf{t}' \times \mathbf{a})$ *are finite, and*

$$\hat{\mathbf{t}} \times' (\mathbf{t}' \times \mathbf{a}) \geq \mathbf{a}$$

**Lemma 6.5. (The Dual Division Algorithm)** *Let* $\mathbf{a},\mathbf{t}$ *satisfy the hypothesis of Lemma* 6.4. *Then for* $\mathbf{q} = \mathbf{t}' \times \mathbf{a}$, *and* $\mathbf{r}$ *defined by*

$$\mathbf{r}_i = \begin{cases} \mathbf{a}_i & \text{if } \mathbf{a}_i < [\hat{\mathbf{t}} \times' (\mathbf{t}' \times \mathbf{a})]_i \\ +\infty & \text{if } \mathbf{a}_i = [\hat{\mathbf{t}} \times' (\mathbf{t}' \times \mathbf{a})]_i \end{cases}$$

*we have*

$$\mathbf{a} = (\hat{\mathbf{t}} \times' \mathbf{q}) \wedge \mathbf{r}.$$

## <u>6.2. An Image Algebra Division Algorithm</u>

Using the isomorphism $\Psi$, we can express these ideas in the image algebra. Let $\hat{\mathbf{t}} \equiv (\mathbf{t}^*)'$ for $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$.

**Lemma 6.6.** *Let* $\mathbf{a} \in \mathbf{F}^{\mathbf{X}}$, $\mathbf{t} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *be such that* $\mathcal{S}_{-\infty}(\mathbf{t}_{\mathbf{x}}) \neq \emptyset \; \forall \; \mathbf{x} \in \mathbf{X}$. *Then each of* $\mathbf{a} \boxed{\wedge} \hat{\mathbf{t}}$ *and* $(\mathbf{a} \boxed{\wedge} \hat{\mathbf{t}}) \boxed{\vee} \mathbf{t}'$ *are finite, and* $\mathbf{a} \geq (\mathbf{a} \boxed{\wedge} \dot{\mathbf{t}}) \boxed{\vee} \mathbf{t}'$.

This next theorem is the counterpart to Lemma 6.3.

**Lemma 6.7.** *Let* $\mathbf{a},\mathbf{b} \in \mathbf{F}^{\mathbf{X}}$. *Then the image* $\mathbf{c} = \mathbf{a} - \mathbf{b}$ *may be expressed using a template in the following way. Define* $\mathbf{s} \in (\mathbf{F}_{-\infty}^{\mathbf{X}})^{\mathbf{X}}$ *by*

$$\mathbf{s}_\mathbf{y}(\mathbf{x}) = \begin{cases} -\mathbf{b}(\mathbf{y}) & \text{if } \mathbf{x} = \mathbf{y} \\ -\infty & \text{otherwise} \end{cases}.$$

*Then* $\mathbf{a} \boxed{\vee}\ \mathbf{s} = \mathbf{a} - \mathbf{b}$.

Using the lattice characteristic function, it is sometimes the case that we can stay within the lattice operations of $\vee$ and $\boxed{\vee}$ and the image algebra operation of $+$ when needing to express a characteristic function. An example of this follows immediately.

**Theorem 6.8. The Division Algorithm.** *Let* $\mathbf{a}$, $\mathbf{t}$ *satisfy the hypothesis of Lemma* 6.6. *Then for* $\mathbf{q} = \mathbf{a} \boxed{\wedge}\ \hat{\mathbf{t}}$ *and* $\mathbf{r}$ *defined by*

$$\mathbf{r} = \mathbf{a} + \chi^{\infty}_{>0}(\mathbf{a})[\mathbf{a} - ((\mathbf{a} \boxed{\wedge}\ \hat{\mathbf{t}}) \boxed{\vee}\ \mathbf{t}')]$$

*we have that*

$$\mathbf{a} = (\mathbf{q} \boxed{\vee}\ \mathbf{t}') \vee \mathbf{r}.$$

**Proof:** We need to show that $\Psi^{-1}(\mathbf{r})$ matches with our definition of the matrix $\mathbf{r}$ in Theorem 6.2. Let $\mathbf{b} = (\mathbf{a} \boxed{\wedge}\ \hat{\mathbf{t}}) \boxed{\vee}\ \mathbf{t}'$. Then using Lemma 6.7, $\mathbf{a}$ - $\mathbf{b} = \mathbf{a} \boxed{\vee}\ \mathbf{s}$, where

$$\mathbf{s}_\mathbf{y}(\mathbf{x}) = \begin{cases} -\mathbf{b}(\mathbf{y}) & \text{if } \mathbf{x} = \mathbf{y} \\ -\infty & \text{otherwise} \end{cases}.$$

Thus, $\mathbf{a} - \mathbf{b} \geq \mathbf{0}$ implies that $\mathbf{a} \boxed{\vee}\ \mathbf{s} \geq \mathbf{0}$. Now,

$$\chi^{\infty}_{>0}(\mathbf{a} \boxed{\vee}\ \mathbf{s}) = \mathbf{c} \in \mathbf{F}^{\mathbf{X}}, \quad \text{where} \quad \mathbf{c}(\mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{a}(\mathbf{x}) > \mathbf{b}(\mathbf{x}) \\ -\infty & \text{if } \mathbf{a}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) \end{cases}.$$

Thus, at location $\mathbf{x} \in \mathbf{X}$, the image $\mathbf{r} = \mathbf{a} + \chi^{\infty}_{>0}[\mathbf{a} \boxed{\vee}\ \mathbf{s}]$ has the gray value

$$\mathbf{r}(\mathbf{x}) = \mathbf{a}(\mathbf{x}) + \mathbf{c}(\mathbf{x}) = \begin{cases} \mathbf{a}(\mathbf{x}) + 0 & \text{if } \mathbf{a}(\mathbf{x}) > \mathbf{b}(\mathbf{x}) \\ \mathbf{a}(\mathbf{x}) + -\infty & \text{if } \mathbf{a}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) \end{cases} = \begin{cases} \mathbf{a}(\mathbf{x}) & \text{if } \mathbf{a}(\mathbf{x}) > \mathbf{b}(\mathbf{x}) \\ -\infty & \text{if } \mathbf{a}(\mathbf{x}) = \mathbf{b}(\mathbf{x}) \end{cases}$$

Under $\Psi$, this remainder image is the same as the vector $\mathbf{r}$ in Lemma 6.2.

Q.E.D.

Iterating k times on an image **a** and a template **t** satisfying the hypothesis of Lemma 6.6, we obtain

$$\mathbf{a} = \bigvee_{i=1}^{k} [(\mathbf{r}^{i} \boxed{\vee} (\mathbf{t}')^{i}] \vee [(\mathbf{a}^{k+1} \boxed{\vee} (\mathbf{t}')^{k+1}]$$

where any template **t** raised to the zero-th power, $\mathbf{t}^0$, is the identity template, **e**.

In the boolean case, there exists an integer m such that

$$\mathbf{a}_{m} \boxed{\vee} (\mathbf{t}')^{m} = 0$$

so that the expression for **a** becomes

$$\mathbf{a} = \bigvee_{k=0}^{m} \mathbf{r}_{k} \boxed{\vee} (\mathbf{t}')^{k}$$

One useful application of this result is in data compression. By encoding the $\mathbf{r}_i$'s in run length code, the image can be represented by fewer bits of data, and reconstructed exactly once **t** is known.

We have the dual Division Algorithm stated in the image algebra also.

**Proposition 6.9.** *Let* $\mathbf{a} \in \mathbf{R}^{\mathbf{X}}$, $\mathbf{t} \in (\mathbf{R}_{+\infty}^{\mathbf{x}})^{\mathbf{X}}$ *be such that* $S_{-\infty}(\mathbf{t}_{\mathbf{x}}) \neq \emptyset \; \forall \, \mathbf{x} \in \mathbf{X}$. *For* $\hat{\mathbf{t}}$ by $\hat{\mathbf{t}} \equiv (\mathbf{t}^{*})'$, *we have that each of* $\mathbf{a} \boxed{\vee} \mathbf{t}'$ *and* $(\mathbf{a} \boxed{\vee} \mathbf{t}') \boxed{\wedge} \hat{\mathbf{t}}$ *are finite, and* $\mathbf{a} \leq (\mathbf{a} \boxed{\vee} \mathbf{t}') \boxed{\wedge} \hat{\mathbf{t}}$.

**Proposition 6.10. (The Dual Division Algorithm).** *Let* **a**, **t** *satisfy the hypothesis of Proposition* 6.9. *Then for* $\mathbf{q} = \mathbf{a} \boxed{\vee} \mathbf{t}'$ *and* **r** *defined by*

$$\mathbf{r} = \mathbf{a} \wedge \chi_{<0}^{\infty}[\mathbf{a} - ((\mathbf{a} \boxed{\vee} \mathbf{t}') \boxed{\wedge} \hat{\mathbf{t}})]$$

*we have that*

$$\mathbf{a} = (\mathbf{q} \boxed{\wedge} \hat{\mathbf{t}}) \wedge \mathbf{r}.$$

CHAPTER 7
TWO EXAMPLES

## 7.1. An Operations Research Problem Stated in Image Algebra Notation

This section gives a short description of the *transportation problem* in linear programming [63], and provides a translation of the dual transportation problem into image algebra notation. Thus, this provides an example of the use of the isomorphism $\Psi^{-1}$.

Let m producers and n consumers of some commodity be given. Let $p_i$ denote the production capacity of producer i, $d_j$ denote the demand of consumer j, and $c_{ij}$ denote the cost of transporting one unit of commodity from producer i to consumer j. The problem is to determine how much commodity to ship from each producer to each consumer so that consumer demands are met, production capacities are not exceeded, and transportation costs are minimized. This can be formulated as a linear programming (LP) problem, which we state as follows.

Let $z_{ij}$ be the number of units of commodity to be shipped from producer i to consumer j. Then the total transportation cost

$$\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} z_{ij}$$

is to be minimized. To stay within production capacity, we also must have

$$\sum_{j=1}^{n} z_{ij} \leq p_i\,,\ i = 1,\ldots,m$$

and to satisfy consumer demands we must have

$$\sum_{i=1}^{m} z_{ij} \geq d_j \, , \; j = 1,\ldots,n \, .$$

Thus the LP problem is to

$$\begin{array}{lll}
\textbf{Minimize} & \displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} c_{ij} z_{ij} & \\
\textbf{Subject to} & -\displaystyle\sum_{j=1}^{n} z_{ij} \geq -p_i \, , \quad i = 1,\ldots,m & (7\text{-}1) \\
 & \displaystyle\sum_{i=1}^{m} z_{ij} \geq d_j \, , \quad j = 1,\ldots,n \, , \text{ and} & (7\text{-}2) \\
 & z_{ij} \geq 0 \quad \text{for all } i,j. &
\end{array}$$

Let $x_i$ be the dual variable associated with the i-th constraint in (7-1), and $y_j$ the dual variable associated with the j-th constraint in (7-2). Then the dual transportation problem is [63]

$$\begin{array}{lll}
\textbf{Maximize} & -\displaystyle\sum_{i=1}^{m} p_i x_i + \sum_{j=1}^{n} d_j y_j & \\
\textbf{Subject to} & -x_i + y_j \leq c_{ij} & \text{for all } i,j \\
 & x_i \geq 0, \; y_j \geq 0 & \text{for all } i,j.
\end{array}$$

This is equivalent to solving

$$\begin{array}{lll}
\textbf{Minimize} & -[-\displaystyle\sum_{i=1}^{m} p_i x_i + \sum_{j=1}^{n} d_j y_j] & \\
\textbf{Subject to} & -x_i + y_j \leq c_{ij} & \text{for all } i,j \\
 & x_i \geq 0, \; y_j \geq 0 & \text{for all } i,j,
\end{array}$$

which is

$$\begin{array}{lll}
\textbf{Minimize} & \sum_{i=1}^{m} p_i x_i - \sum_{j=1}^{n} d_j y_j & \\
\textbf{Subject to} & -x_i + y_j \leq c_{ij} & \text{for all i,j} \\
 & x_i \geq 0,\ y_j \geq 0 & \text{for all i,j.}
\end{array}$$

Make a change of variables by letting

$$v_j = -y_j \quad \text{and } u_i = -x_i, \text{ for all i,j}.$$

Then we have the equivalent dual LP problem

$$\begin{array}{lll}
\textbf{Minimize} & \sum_{j=1}^{n} d_j v_j - \sum_{i=1}^{m} p_i u_i & \\
\textbf{Subject to} & u_i - v_j \leq c_{ij} & \text{for all i,j} \\
 & u_i \leq 0,\ v_j \leq 0 & \text{for all i,j.}
\end{array}$$

Using the theory of *complementary slacks* [63], if we assume that the producers $p_i$ have value $p_i > 0$ for all i, then we can be guaranteed that for each $i = 1,\ldots,m$, there exists at least one $j \in \{1,\ldots,n\}$ such that

$$u_i - v_j = c_{ij},$$

and, hence,

$$u_i = \min_{j=1}^{n} \{c_{ij} + y_j\}, \tag{7-3}$$

where $\mathbf{u} = (u_1,\ldots,u_m)$ and $\mathbf{v} = (v_1,\ldots,v_n)$ are optimal feasible solutions. We can rewrite (7-3) in vector notation, as

$$\mathbf{u} = C \times' \mathbf{v},$$

and $u_i$, $v_j \leq 0$.

To formulate this problem in context of the image algebra, we define $\mathbf{X}$ and $\mathbf{Y}$ to be non-empty, finite coordinate sets, $|\mathbf{X}| = m$, $|\mathbf{Y}| = n$. Define $\mathbf{d} \in (\mathbf{R}_{\pm\infty}^{\mathbf{Y}})^{\mathbf{Y}}$ by

$$\mathbf{d}_{\mathbf{y}_i}(\mathbf{y}_j) = \begin{cases} d_i & i = j \\ -\infty & \text{otherwise} \end{cases},$$

and define $\mathbf{p} \in (\mathbf{R}_{\pm\infty}^{\mathbf{X}})^{\mathbf{X}}$ by

$$\mathbf{p}_{\mathbf{x}_i}(\mathbf{x}_j) = \begin{cases} p_i & i = j \\ -\infty & \text{otherwise} \end{cases}.$$

Define $\mathbf{a} \in \mathbf{R}_{\pm\infty}^{\mathbf{X}}$, $\mathbf{b} \in \mathbf{R}_{\pm\infty}^{\mathbf{Y}}$ by

$$\mathbf{a}(\mathbf{x}_i) = u_i \text{ (the variable)}$$

$$\mathbf{b}(\mathbf{x}_i) = v_i \text{ (the variable)}.$$

Then we have:

| LP | Image Algebra |
|---|---|
| $\sum_{j=1}^{n} d_j v_j$ | $\sum(\mathbf{b} \boxed{\vee} \mathbf{d})$ |
| $\sum_{i=1}^{m} p_i u_i$ | $\sum(\mathbf{a} \boxed{\vee} \mathbf{p})$ |

Now, define $\mathbf{t} \in (\mathbf{R}_{\pm\infty}^{\mathbf{Y}})^{\mathbf{X}}$ by

$$\mathbf{t}_{\mathbf{x}_i}(\mathbf{y}_j) = c_{ij}$$

We have the equation $\mathbf{u} = C \times' \mathbf{v}$ translates as $\mathbf{a} = \mathbf{b} \boxed{\wedge} \mathbf{t}$. Thus, in image algebra notation, the dual LP problem is

| | |
|---|---|
| **Minimize** | $\sum(\mathbf{b} \boxed{\vee} \mathbf{d}) - \sum(\mathbf{a} \boxed{\vee} \mathbf{p})$ |
| **Subject to** | $\mathbf{a} = \mathbf{b} \boxed{\wedge} \mathbf{t}$ |
| | $\mathbf{a} \leq \mathbf{0},\ \mathbf{b} \leq \mathbf{0}$ |

## 7.2. An Image Complexity Measure

This section presents an *image complexity measure*, a term used in image processing to describe any method which provides a quantitative measure of some feature or set of features in an image. Image complexity measures are used either as a pre-processing step in which the measures help direct the selection of the next processing step, or in conjuction with other information derived from the image to identify objects of interest.

The measure investigated [64] is based on a method discussed by Mandelbrot [65] for curve length measurements. The original algorithm was modified and translated into image algebra. The measure itself consists of a graph which in theory gives an indication of the rate of change of variation in the gray level surface. The algorithm for computing the measure is presented, followed by a discussion of an application to 12 outdoor images.

The general approach of the algorithm is to make successive approximations of the area of a gray level surface, and then plot the approximations using a log-log scale. The log-log scale is purported to allow a better visual inspection of the information contained in the graph.

Consider all points with distance to the gray level surface of no more than k. These points form a blanket of thickness 2k, and the suggested surface area A(k) of the gray level surface is the volume of the blanket divided by 2k. Here we have A(k) increasing as k decreases.

To begin the computation of the surface area for k =1,2,..., an upper surface $\mathbf{u}_k$ and a lower surface $\mathbf{b}_k$ are defined iteratively in the following manner. Let $\mathbf{a}$ be the input image. Let

$$\mathbf{u}_o = \mathbf{a}, \quad \mathbf{b}_o = \mathbf{a}$$

Then define $\mathbf{u}_k$ and $\mathbf{b}_k$ for k = 1,2,..., by

$$\mathbf{u}_k = \mathbf{u}_{k-1} \boxed{\vee}\ \mathbf{t}$$

$$\mathbf{b}_k = \mathbf{b}_{k-1} \boxed{\wedge}\ -\mathbf{t}$$

where

$$\mathbf{t} = \begin{array}{|c|c|c|} \hline & 0 & \\ \hline 0 & 1 & 0 \\ \hline & 0 & \\ \hline \end{array}$$

The volume v(k) of the "blanket" between the upper and lower surfaces is calculated for each k by computing

$$\mathbf{p}_1(k) = \mathbf{u}_k \oplus \mathbf{s}, \qquad \mathbf{q}_1(k) = \mathbf{b}_k \oplus (-\mathbf{s})$$

where

$$\mathbf{s} = \begin{array}{|c|c|} \hline 0.67 & 0.33 \\ \hline 0.33 & 0.67 \\ \hline \end{array}$$

Let $v_1(k) = \sum[\mathbf{p}_1(k) + \mathbf{q}_1(k)]$.

This method of estimating the volume was derived using elementary calculus. We explain the method for calculating the volume between the upper surface and the coordinate set $\mathbf{X}$. The volume between the lower surface and $\mathbf{X}$ is found in a similar way. Given four pixel locations in $\mathbf{X}$, (i,j), (i,j+1), (i+1,j), and (i+1,j+1), a box was constructed from the eight points in $\mathbf{R}^3$ corresponding to the four gray values $\mathbf{u}_k(i,j)$, $\mathbf{u}_k(i,j+1)$, $\mathbf{u}_k(i+1,j)$, $\mathbf{u}_k(i+1,j+1)$, and the four given pixels. Drawing a line from $\mathbf{u}_k(i,j)$ to $\mathbf{u}_k(i+1,j+1)$ and a line from (i,j) to (i+1,j+1), the volume of the triangular column determined by the six points $\mathbf{u}_k(i,j)$, $\mathbf{u}_k(i+1,j)$, $\mathbf{u}_k(i+1,j+1)$, (i,j), (i+1,j), and (i+1,j+1) was found using elementary methods from calculus. Similarly, the volume of the triangular column determined by the six points $\mathbf{u}_k(i,j)$, $\mathbf{u}_k(i,j+1)$, $\mathbf{u}_k(i+1,j+1)$, (i,j), (i,j+1), and (i+1,j+1) was determined. The volume

of the two pieces are added together to give an estimate to the volume of the box determined by the eight initial points. This is done over the entire coordinate set $\mathbf{X}$, and all volumes added together to give an estimate of the volume between $\mathbf{X}$ and the gray value surface $\mathbf{u}_k$. The method was expressed using the image algebra operation $\oplus$ and an invariant template, omitting the boundary effects. Using this approach, the volume is overestimated, so it is corrected by applying a variant template $\mathbf{w}$ effective only on the edge pixels. Define $\mathbf{w}$ by

$$\mathbf{w}_x = \begin{array}{|c|c|}\hline 0.33 & 0.67 \\ \hline \end{array}$$

if $\mathbf{x}$ is a top edge pixel and not the top right corner pixel,

$$\mathbf{w}_x = \begin{array}{|c|}\hline 0.33 \\ \hline \end{array}$$

if $\mathbf{x}$ is the top right corner pixel,

$$\mathbf{w}_x = \begin{array}{|c|}\hline 0.67 \\ \hline 0.33 \\ \hline \end{array}$$

if $\mathbf{x}$ is a right edge pixel but not the top right corner pixel, and $\mathbf{w}_x = \mathbf{0}$, if $\mathbf{x}$ is otherwise.

To correct for the extra volume added in on the edge pixels, we calculate

$$\mathbf{p}_2(k) = \mathbf{u}_k \oplus (-\mathbf{w}), \qquad \mathbf{q}_2(k) = \mathbf{b}_k \oplus \mathbf{w}$$

and let $\text{volerr}(k) = \sum [\mathbf{p}_2(k) + \mathbf{q}_2(k)]$. The correct volume $v(k)$ is

$$v(k) = v_1(k) + \text{volerr}(k).$$

The approximated surface area is

$$\text{area}(k) = \frac{v(k)}{2k}.$$

The rate of change of log(area(k)) with respect to log(k) contains important information about the image. The slope S(k) of area(k) versus k is computed on a log-log scale for each k

by finding the best fitting straight line through the three points

$$(\log(k-1), \log(\text{area}(k-1))),\ (\log(k), \log(\text{area}(k))),\ (\log(k+1), \log(\text{area}(k+1))).$$

The graph of S(k) versus k is called the *signature* of the image. We can also calculate a signature for the case where the array $\mathbf{X}$ represents the bottom surface and $\mathbf{u}_k$ the upper surface. We call this the *upper signature.* Similarly, the signature which is calculated using $\{\mathbf{b}_k\}$ for the lower surfaces and $\mathbf{X}$ for the upper surfaces is called the*lower signature.*

This algorithm was run on 12 images. For each image, we calculated the input image, $\mathbf{u}_i$, $\mathbf{b}_i$, $i = 1,...,50$, and the graph of the upper and lower signatures.

As k increases, regions of pixels initially having the highest gray values decrease in size in the images $\mathbf{b}_k$. However, as k increases, the images $\mathbf{u}_k$ shrink regions having lower gray values. In theory, this asymmetry can be taken advantage of. Roughly, the lower signature represents the shape of objects with high gray values, and the upper signature represents the distribution of objects throughout the image. The images to which we applied this method were infrared, so we were mainly interested in the lower signatures.

The magnitude of the curve S(k) is related to the information lost on objects with details less than k in size. The more gray level variation at distance k, the higher the values for S(k). Thus, if at small k S(k) is large, then there is "high-frequency" gray level variations, and if at large k S(k) is large, then we have "low-frequency" gray level variations. The curve S(k) thus gives us information about the rate of change of variations in the gray level surface.

After running the program on a dozen images, we have concluded that this algorithm is too sensitive to the great variance in outdoor scenery. For example, an image which has a background of trees and no targets, and an image which has two distinct targets and no trees as background have similar graphs for the lower signatures. While the lower signature

represents more of the shape of the hot objects (areas with high gray values) in the image, in one image we have no hot objects while in the other, there are two distinct hot objects. As another example, in two other images we have a target with a road and a field as background, yet the graphs for the upper signatures for these images have a very distinct difference. The theory suggests that upper signatures should represent similar targets, but we cannot draw that conclusion from this data. The examples given in the original paper [64] are of a very regular texture and are presented in a controlled environment. It is very likely that the controlled environment in which the data was taken is one reason why this algorithm was successful for those authors.

The initial goal of investigating this type of complexity measure was that these graphs would hopefully give a measure of gray level variation within an image and help in choosing a more effective edge operator. If an image has a high incidence of gray level variation at small values of k, then it is reasonable to assume a more sensitive mask, such as the gradient mask, would give better results. Otherwise, if an image had small values of S(k) at small values of k, then computation time could be saved by using a Sobel operator instead of a computationally intensive edge operator such as the Kirsch. Unfortunately, the algorithm did not produce data that leads to this conclusion.

# CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

We have shown that the establishment of an isomorphism between a lattice-based matrix theory and the image algebra provides a powerful tool in image processing which was not available previous to this research. Just as linear transforms are able to be represented as matrices within the structure of linear algebra, lattice transforms are able to be represented as matrices within the structure of the minimax algebra, and thus all mathematical results of the minimax algebra are applicable to solving problems in image processing.

In particular, we have shown that

(1) Many notions encountered in investigating linear transforms, such as the eigenvalue/eigenvector problem, rank, linear independence, and solutions to systems of equations, have their counterpart in image algebra. The use of these notions to solving specific image processing problems remains to be seen.

(2) The image algebra is useful as a model in mapping a class of non-linear transforms to parallel architectures. It is feasible to map any arbitrary lattice transform to most parallel architectures; that is, given a network of processors that are interconnected by communication links, a lattice transform has a weak decomposition into a product of lattice transforms that are each local with respect to the network in and only if every pair of processors has a two-way path of communication between them. There is at most one weak (pointwise) operation of maximum, with the remaining operations being convolutions. The pointwise maximum which represents the fact that a small amount of storage is needed to compute the transform is not an unreasonable restriction.

(3) Mathematical morphology is a special subclass of the lattice transforms, namely the transforms corresponding to invariant templates from $\mathbf{X}$ to $\mathbf{X}$ having the target pixel as part of their support at each pixel. Thus, the lattice subalgebra of the image algebra generalizes mathematical morphology. The image algebra is a clear and simple mathematical representation of morphological concepts, avoiding the cumbersome notion of *umbra*, for example.

(4) The minimax algebra lends itself well to the investigation of matrix decomposition techniques. A weak decomposition was shown for any matrix lattice transform. Thus, the application of a matrix product for any purpose can be implemented in parallel. This includes problems from operations research.

(5) Although the minimax algebra is not a Euclidean domain, the representation of a type of division algorithm was found. In boolean images, there are clear applications of these types of skeletonizing techniques to image compression.

We now list some suggestions for further research. The author continues to pursue this general area of research, and would appreciate being informed of any results obtained related to the following problems.

Unlike linear algebra, the minimax algebra is relatively unknown. The matrix properties developed so far in the literature [38] were meant primarily for use in operations research problems. A wealth of mathematical results, with possible applications to real world problems, are obviously obtainable from investigating other uses of the minimax algebra. The matrix decomposition techniques presented in this dissertation are a clear example. Specific problems are: investigate Chapter 3's statements for applications to solving image processing problems. What specific applications does the eigenproblem have in image processing? The statements in Chapter 3 are only a few of the minimax algebra properties that were mapped

to image algebra. Investigate other properties in the *Minimax Algebra* book which may have uses in image processing. Is there a minimax transform equivalent to the linear Fourier transform? What are its uses? What other types of linear transforms have a minimax equivalent? Can other techniques in linear algebra, such as the SVD, Kronecker products, etc. be described in a similar way in the minimax algebra? Develop techniques for decomposition of square matrices that are block toeplitz with toeplitz blocks, with non-null diagonal entries. As mentioned in Chapter 4, this type of matrix corresponds to the concept of a structuring element in mathematical morphology, and to a class of invariant templates in the image algebra.

Another question concerns the existence theorem, Theorem 5.24. Does there exist a local decomposition that is not weak, that is, a decomposition with no operation of $\vee$? This would relieve the storage requirement associated with $\vee$. Also, is there a similar existence theorem with the digraph D(W) replacing the graph G(W)? Although the existence theorem given in Chapter 5 is constructive, one would rarely implement the decomposition in this manner. Describe a general decomposition technique giving local templates and which is minimal with respect to some criterion, such as the number of factors in the decomposition. Also, what type of applications to operations research does the decomposition theorem have?

The continuous cases for the image algebra operands $\oplus$, $\boxed{\vee}$, and $\circledvee$ have yet to be well established. Establish the mathematical foundations for the continuous counterparts involving the lattice operations $\boxed{\vee}$, $\circledvee$, and $\vee$.

In his dissertation Gader asks the question about relating circulant templates under $\boxed{\vee}$ or $\circledvee$ to group algebras. Will the formal minimax matrix relationship established with the image algebra make the pursuit of this particular question any easier?

Is there a practical use for the division algorithm? Generalize the division algorithm to a sequence of templates $\mathbf{t}_i$ replacing the single $\mathbf{t}$.

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## BIOGRAPHICAL SKETCH

Jennifer L. Davidson was born in Dayton, Ohio, and grew up in the Washington, D.C. area. She received the B.A. in physics from Mount Holyoke College, Massachusetts, and the M.S. in mathematics from the University of Florida in 1986. Her research interests include applied mathematics, image processing, and computer vision, and she is the author of over 17 research papers including a journal publication.

Ms. Davidson is a member of the IEEE Computer Society, Society for Photo-Optical Instrumentation Engineers, Society of Industrial and Applied Mathematics, Mathematical Association of America, and the American Mathematical Society.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Gerhard X. Ritter

Gerhard X. Ritter, Chairman
Professor of Mathematics
Professor of Computer and Information Science

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Beverly L. Brechner
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

James E. Keesling
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Li-Chien Shen
Associate Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

David C. Wilson

David C. Wilson
Professor of Mathematics

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Joseph N. Wilson
Assistant Professor of Computer and Information Sciences

This dissertation was submitted to the Graduate Faculty of the Department of Mathematics in the College of Liberal Arts and Sciences and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

August, 1989

Dean, Graduate School